

Tensor products of modal logics

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Products were introduced in the 1970s as a natural type of combined modal logics. They arise in different areas of pure and applied logics — spatial reasoning, multi-agent systems, quantified modal and intuitionistic logics etc. The theory of products was systematized and essentially developed first in the paper [GS98] and then in the book [GKWZ03], but during the past 10 years new important results were proved and the research is going on, cf. [Kur07].

Recall that the product of modal logics is defined as the logic of the class of products of their Kripke frames. On the one hand, this definition is quite natural, and in some cases products can be simply axiomatized and have nice properties. On the other hand, products are always Kripke complete. However, Kripke semantics sometimes may be inadequate. So different logics L_1, L'_1 can have the same frames; then $L_1 \times L_2 = L'_1 \times L_2$ for any L_2 — which looks strange. Another peculiarity is logical non-invariance: it may happen that for some frames $\text{Log}(F) = \text{Log}(F')$, while $\text{Log}(F \times G) \neq \text{Log}(F' \times G)$. Also, if logics L_1 and L_2 are consistent, but L_1 has an empty class of frames, then $L_1 \times L_2$ is inconsistent.

To amend the situation, we can try to define products of Kripke of models (or, equivalently, general Kripke frames or modal algebras). The following problem was mentioned in [Kur07, p. 877]: *There are several attempts for extending the product construction from Kripke complete logics to arbitrary modal logics, mainly by considering product-like constructions on Kripke models. All the suggested methods so far result in sets of formulas that are not closed under the rule of Substitution.*

Nevertheless, by that time a possible answer already existed (but was unnoticed): it was given by Yasusi Hasimoto in [Has00], who introduced so-called *shifted products of modal algebras*. In this paper we show that a shifted product acts on Boolean algebras exactly as a tensor product (an observation first made by Dov Gabbay [GSS]). So now we call this operation a *tensor product of modal algebras*. We also propose a filtration technique for models based on tensor products and obtain some decidability results.

1 Tensor products and chequered valuations

It is well known that we can regard every Boolean algebra as a Boolean ring. Every Boolean ring is a commutative algebra with an idempotent multiplication over the two-element field \mathbf{F}_2 . So the standard tensor product construction is applicable here [AM69].

Viz., the tensor product of algebras A, B is the pair $(\pi, A \otimes B)$, where $\pi : (a, b) \mapsto a \otimes b$ is a bilinear map $A \times B \rightarrow A \otimes B$ such that every bilinear $f : A \times B \rightarrow C$ uniquely factors through π , i.e., $f = g \cdot \pi$ for a unique linear g . The multiplication in $A \otimes B$ is defined in such a way that $(a \otimes b)(c \otimes d) = ac \otimes bd$.

Proposition 1.1. *The tensor product of Boolean rings is a Boolean ring.*

Our aim is to define tensor products of modal algebras. To this end, let us describe the tensor product construction for Boolean rings more explicitly.

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Definition 1.2. A set $U \times V$, where $U \subseteq X$, $V \subseteq Y$, is called a *rectangle* in $X \times Y$. A *chequered subset* of $X \times Y$ is a finite union of rectangles.

Proposition 1.3. [Has00] *The set of all chequered subsets of $W_1 \times W_2$ is closed under Boolean operations. Moreover, if A_i is a subalgebra of 2^{W_i} , $i = 1, 2$, then the set of all finite unions of rectangles $V_1 \times V_2$, where $V_i \in A_i$, is closed under Boolean operations as well.*

For nonempty sets X, Y let $ch(X, Y)$ be the Boolean algebra of all chequered subsets of $X \times Y$.

Theorem 1.4. [GSS] *Let X, Y be nonempty sets, A, B subalgebras of $2^X, 2^Y$ respectively, C the Boolean algebra of finite unions of rectangles $U \times V$, where $U \in A, V \in B$. Consider the map $\pi : 2^X \times 2^Y \rightarrow ch(X, Y)$ such that $\pi(U, V) = U \times V$. Then $(\pi|(A \times B), C)$ is the tensor product of Boolean algebras A and B . In particular, $(\pi, ch(X, Y))$ is the tensor product of 2^X and 2^Y .*

Proposition 1.5. [Has00] *Consider Kripke frames $F_1 = (W_1, R_1)$, $F_2 = (W_2, R_2)$ and their product [GS98] $F_1 \times F_2 = (W_1 \times W_2, R_1^\times, R_2^\times)$. Then*

(1) *for any rectangle $U \times V$ we have*

$$R_1^{\times -1}(U \times V) = R_1^{-1}(U) \times V, \quad R_2^{\times -1}(U \times V) = U \times R_2^{-1}(V);$$

(2) *if (F_1, A_1) and (F_2, A_2) are general 1-frames, then $(F_1 \times F_2, A_1 \otimes A_2)$ is a general 2-frame. In particular, $(F_1 \times F_2, ch(W_1, W_2))$ is a general 2-frame.*

Definition 1.6. The frame $(F_1 \times F_2, A_1 \otimes A_2)$ is called the *tensor product of general frames* (F_1, A_1) and (F_2, A_2) :

$$(F_1, A_1) \otimes (F_2, A_2) := (F_1 \times F_2, A_1 \otimes A_2).$$

In particular, we have the *tensor product of Kripke frames* $F_1 \otimes F_2 = (F_1 \times F_2, ch(W_1, W_2))$. Tensor products of Kripke frames are also called *chequered frames*.

Theorem 1.7. [GSS] *If (A, \diamond_1) , (B, \diamond_2) are normal 1-modal algebras, then there exists a unique 2-modal algebra structure on $A \otimes B$ such that for any $a \in A, b \in B$*

$$\diamond_1^\times(a \otimes b) = \diamond_1 a \otimes b, \quad \diamond_2^\times(a \otimes b) = a \otimes \diamond_2 b.$$

Proof. Due to the Jónsson–Tarski representation theorem [CZ96], this follows from Proposition 1.5; the operations \diamond_1^\times and \diamond_2^\times are introduced according to (1). \square

For classes of algebras (general frames) $\mathfrak{A}, \mathfrak{B}$, put $\mathfrak{A} \otimes \mathfrak{B} := \{A \otimes B \mid A \in \mathfrak{A}, B \in \mathfrak{B}\}$.

Definition 1.8. The *tensor product* of logics L_1 and L_2 is the logic

$$L_1 \otimes L_2 := \text{Log}(\text{Alg}(L_1) \otimes \text{Alg}(L_2)).$$

Clearly, $L_1 \otimes L_2 = \text{Log}(\text{GFr}(L_1) \otimes \text{GFr}(L_2))$.

The next proposition easily follows from this definition.

Proposition 1.9. 1. $L_1 \otimes L_2$ *is consistent iff L_1 and L_2 are consistent.*

2. *If L_1 and L_2 are consistent, then $L_1 \otimes L_2$ is conservative over L_1 and L_2 .*

3. *If $L_1 \otimes L_2$ is consistent and Kripke complete, then L_1 and L_2 are Kripke complete.*

2 Logical invariance

This section contains some important basic properties of tensor products. The results of this section were obtained in [Has00].

Theorem 2.1. *Let $\mathfrak{A}, \mathfrak{A}', \mathfrak{B}, \mathfrak{B}'$ be classes of 1-modal algebras, and $\text{Log}(\mathfrak{A}) = \text{Log}(\mathfrak{A}')$, $\text{Log}(\mathfrak{B}) = \text{Log}(\mathfrak{B}')$. Then $\text{Log}(\mathfrak{A} \otimes \mathfrak{B}) = \text{Log}(\mathfrak{A}' \otimes \mathfrak{B}')$.*

Corollary 2.2. *Let $\mathfrak{C}_1, \mathfrak{C}_2$ be classes of 1-modal algebras or general frames, $L_1 = \text{Log}(\mathfrak{C}_1)$, $L_2 = \text{Log}(\mathfrak{C}_2)$. Then $L_1 \otimes L_2 = \text{Log}(\mathfrak{C}_1 \otimes \mathfrak{C}_2)$. In particular, if L_1 and L_2 are Kripke complete then for any classes of Kripke frames $\mathfrak{F}_1, \mathfrak{F}_2$ such that $L_i = \text{Log}(\mathfrak{F}_i)$, $i = 1, 2$, we have that $L_1 \otimes L_2 = \text{Log}(\mathfrak{F}_1 \otimes \mathfrak{F}_2)$ is a logic of a class of chequered frames.*

For classes of Kripke frames $\mathfrak{F}, \mathfrak{G}$, $\mathfrak{F} \times \mathfrak{G} := \{F \times G \mid F \in \mathfrak{F}, G \in \mathfrak{G}\}$. For logics L_1, L_2 , $L_1 \times_{fin} L_2 := \text{Log}(\text{Fr}_{fin}(L_1) \times \text{Fr}_{fin}(L_2))$, where $\text{Fr}_{fin}(L)$ denotes the class of all finite L -frames. Recall that L has the *finite model property* (the fmp, for short) if $L = \text{Log}(\text{Fr}_{fin}(L))$. A logic $L_1 \times L_2$ has the *product fmp* if $L_1 \times L_2 = L_1 \times_{fin} L_2$.

Corollary 2.3. *1. For any L_1, L_2 , $L_1 \otimes L_2 = \text{Log}((F_{L_1}, A_{L_1}) \otimes (F_{L_2}, A_{L_2}))$, where F_L denotes the canonical frame of a logic L , and (F_L, A_L) denotes its general canonical frame.*

2. If L_1, L_2 are canonical, then $L_1 \otimes L_2 = \text{Log}(F_{L_1} \otimes F_{L_2})$.

3. If L_1, L_2 are Kripke complete, then $L_1 \times L_2 \subseteq L_1 \otimes L_2 \subseteq L_1 \times_{fin} L_2$.

4. Let L_1, L_2 be Kripke complete logics. $L_1 \times L_2$ has the product fmp iff

$$L_1 \times L_2 = L_1 \otimes L_2 = L_1 \times_{fin} L_2;$$

it follows that if $L_1 \times L_2$ has the product fmp, then for any \mathfrak{F}_i such that $L_i = \text{Log}(\mathfrak{F}_i)$, $i = 1, 2$, we have $L_1 \times L_2 = \text{Log}(\mathfrak{F}_1 \times \mathfrak{F}_2)$.

5. Suppose L_1 and L_2 have the fmp. Then $L_1 \otimes L_2 = L_1 \times_{fin} L_2$, and

$$L_1 \times L_2 \text{ has the product fmp iff } L_1 \times L_2 = L_1 \otimes L_2.$$

It follows that usually modal and tensor products are different. In fact, modal products of logics with the fmp in many cases do not have the product fmp [GKWZ03]. Rare exceptions are $\mathbf{K} \times \mathbf{K}$ and $\mathbf{S5} \times \mathbf{S5}$ [GS98], [GKWZ03]. Further on we describe some other cases.

3 Filtrations of chequered models

Recall the standard construction of *filtration of Kripke models* [CZ96].

Definition 3.1. Let $M = (W, R, \theta)$ be a model, Φ a set of formulas.

Consider the equivalence relation $\sim_\Phi := \{(x, x') \mid \forall \varphi \in \Phi (M, x \models \varphi \Leftrightarrow M, x' \models \varphi)\}$.

$[x]$ denotes the equivalence class of x w.r.t. \sim_Φ . Consider relations $R_{\min}^{M, \Phi}$ and $R_{\max}^{M, \Phi}$ on the quotient set W/\sim_Φ :

$$\begin{aligned} R_{\min}^{M, \Phi} &:= \{([x], [y]) \mid \exists x' \in [x] \exists y' \in [y] \ x' R y'\}, \\ R_{\max}^{M, \Phi} &:= \{([x], [y]) \mid \forall \varphi (\diamond \varphi \in \Phi \& M, y \models \varphi \Rightarrow M, x \models \diamond \varphi)\}. \end{aligned}$$

They are called the *minimal* and the *maximal filtrating relations*. A model $\bar{M} = (W/\sim_\Phi, \bar{R}, \bar{\theta})$ is called a *filtration of M through Φ* , if for any $p \in \Phi$ $\bar{\theta}(p) = \{[x] \mid M, x \models p\}$, and $R_{\min}^{M, \Phi} \subseteq \bar{R} \subseteq R_{\max}^{M, \Phi}$.

For models M_1, M_2 , the notation $M_1, x \sim_{\Phi} M_2, y$ means $\forall \varphi \in \Phi (M_1, x \models \varphi \Leftrightarrow M_2, y \models \varphi)$.

Lemma 3.2 (Filtration Lemma). *[CZ96] Consider a set of formulas Φ closed under subformulas. If \bar{M} is a filtration of M through Φ , then $M, x \sim_{\Phi} \bar{M}, [x]$ for any x in M .*

Definition 3.3. [Gab72] A logic L admits filtration if it is Kripke complete and for any L -frame (W, R) , for any model $M = (W, R, \theta)$, and for any finite set of formulas Ψ closed under subformulas, there exists a filtration $(\bar{W}, \bar{R}, \bar{\eta})$ of M through Ψ such that (\bar{W}, \bar{R}) is an L -frame.

It is well known that many logics admit filtration (for example, **K, K4, T, S4, S5**).

It follows that if a logic L admits filtration, then it has the fmp. Moreover, to check the L -satisfiability of a formula φ , it is sufficient to consider L -frames of cardinality not greater than $2^{\#\varphi}$, where $\#\varphi$ denotes the cardinality of $sub(\varphi)$, $sub(\varphi)$ denotes the set of all subformulas of φ . Our aim is to formulate an analogous result for chequered frames.

Definition 3.4. Consider Kripke frames $F_1, F_2, F_i = (W_i, R_i)$, a model $M = (F_1 \times F_2, \theta)$, and a set of formulas Φ . Consider relations \sim_i on W_i :

$$\begin{aligned} \sim_1 &:= \{(x, x') \mid (x, y) \sim_{\Phi} (x', y) \text{ for all } y \in W_2\}; \\ \sim_2 &:= \{(y, y') \mid (x, y) \sim_{\Phi} (x, y') \text{ for all } x \in W_1\}. \end{aligned}$$

The pair (\sim_1, \sim_2) is called the Φ -granulation of M . Obviously, \sim_i is an equivalence relation on W_i .

Proposition 3.5. *Consider a model $M = ((W_1, R_1) \otimes (W_2, R_2), \theta)$, a finite set of formulas Φ , and the Φ -granulation (\sim_1, \sim_2) . Then the quotient sets W_1/\sim_1 and W_2/\sim_2 are finite.*

Lemma 3.6. *Suppose L admits filtration, F, G are Kripke frames, $F \models L$, φ is true at some point in a chequered model $M = (F \otimes G, \theta)$, (\sim_1, \sim_2) is the $sub(\varphi)$ -granulation of M . Then there exists a Kripke frame \bar{F} such that $\bar{F} \models L$, φ is satisfiable in $\bar{F} \times G$, and the cardinality of \bar{F} is not greater than*

$$2^{\#\varphi \cdot |W_2/\sim_2|}.$$

Unlike the usual filtration technique, this lemma does not estimate the size of a counter-model, since the upper bound depends on W_2/\sim_2 . However, in some cases it implies decidability results for tensor and modal products.

Theorem 3.7. *If L_1 is Kripke complete, L_2 is tabular, then $L_1 \times L_2 = L_1 \otimes L_2$.*

Proof. $L_1 \times L_2$ is complete with respect to the class $\mathfrak{C} = \{F \times G \mid F \models L_1, G \models L_2, G \text{ is rooted}\}$, see e.g. [GKWZ03]. If $G \models L_2$ and G is rooted, then by tabularity of L_2 , G is finite [CZ96]. So all frames in the class \mathfrak{C} are chequered, and $L_1 \times L_2 = L_1 \otimes L_2$. \square

Hence we readily obtain the following properties of modal products with tabular logics.

Corollary 3.8. *For a class of frames \mathfrak{F} , and a finite frame G , $\text{Log}(\mathfrak{F}) \times \text{Log}(G) = \text{Log}(\mathfrak{F} \times \{G\})$.*

Corollary 3.9. *The modal product of tabular logics is tabular: if F and G are finite, then*

$$\text{Log}(F) \times \text{Log}(G) = \text{Log}(F \times G).$$

Theorem 3.10. *Suppose L_1 admits filtration, L_2 is tabular. Then:*

1. $L_1 \times L_2$ has the exponential product fmp;
2. if the finite frame problem is decidable for L_1 , then $L_1 \times L_2$ is decidable.

Proof. Let $L_2 = \text{Log}(G)$ for a finite G of size n . By Lemma 3.6, φ is $L_1 \times L_2$ -satisfiable iff φ is satisfiable in a frame $F \times G$, where $F \models L_1$ and the size of F is not greater than $2^{\#\varphi \cdot n}$. \square

4 Open questions

The above considerations can be easily transferred to the polymodal case. In particular, logical invariance holds for the polymodal case. One of the corollaries is associativity of the tensor products. Note that this question for modal products is open, see e.g. [Kur07, p. 877].

Still there are many open questions about tensor products of modal logics, for example:

1. Does Kripke completeness transfer from L_1 and L_2 to $L_1 \otimes L_2$?
2. Do there exist L_1, L_2 such that $L_1 \times L_2 = L_1 \otimes L_2$, but $L_1 \times L_2$ lacks the product fmp and L_1, L_2 are non-tabular?
3. Do there exist L_1, L_2 such that $L_1 \times L_2$ is undecidable, but $L_1 \otimes L_2$ is decidable?
4. Do there exist L_1, L_2 such that $L_1 \times L_2$ is not finitely axiomatizable, but $L_1 \otimes L_2$ is finitely axiomatizable?
5. Suppose that L_1 is decidable, L_2 is tabular; is $L_1 \otimes L_2 (= L_1 \times L_2)$ decidable?

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