Note on the Odd Perfect Numbers

Frank Vega

July 15, 2022

Note on the Odd Perfect Numbers

Frank Vega

CopSonic, 1471 Route de Saint-Nauphary 82000 Montauban, France

Abstract

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In 2011, Solé and and Planat stated that the Riemann Hypothesis is true if and only if the inequality $\frac{\pi^2}{6} \times \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > e^{\gamma} \times \log \theta(q_n)$ is satisfied for all primes $q_n > 3$, where $\theta(x)$ is the Chebyshev function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. Under the assumption that the Riemann Hypothesis is true and the inequality $\frac{\pi^2}{\beta} \times \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > e^{\gamma} \times \log \theta(q_n)$ is satisfied for infinitely many prime numbers q_n and $\beta \geq 6.0008$, then we prove that there is not any odd perfect number at all.

Keywords: Riemann Hypothesis, Prime numbers, Odd perfect numbers, Superabundant numbers, Sum-of-divisors function
2000 MSC: 11M26, 11A41, 11A25

1. Introduction

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. As usual $\sigma(n)$ is the sum-of-divisors function of n:

$$\sum_{d \mid n} d$$

where $d \mid n$ means the integer d divides n, $d \nmid n$ means the integer d does not divide n and $d^k \parallel n$ means $d^k \mid n$ and $d^{k+1} \nmid n$. Define $f(n)$ and $G(n)$ to be $\frac{\sigma(n)}{n}$ and $\frac{f(n)}{\log \log n}$ respectively, such that log is the natural logarithm. We know these properties from these functions:

Proposition 1.1. *[1]. Let $\prod_{i=1}^{r} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_r$ with natural numbers as exponents $a_1, \ldots, a_r$. Then,*

$$f(n) = \left(\prod_{i=1}^{r} \frac{q_i}{q_i - 1}\right) \times \prod_{i=1}^{r} \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

Email address: `vega.frank@gmail.com` (Frank Vega)

Proposition 1.2. *For every prime power q^a, we have that $f(q^a) = \frac{q^{a+1}-1}{q^a \times (q-1)}$ [2]. If $m, n \geq 2$ are natural numbers, then $f(m \times n) \leq f(m) \times f(n)$ [2]. Moreover, if p is a prime number, and a, b two positive integers, then [2]:*

$$f(p^{a+b}) - f(p^a) \times f(p^b) = -\frac{(p^a - 1) \times (p^b - 1)}{p^{a+b-1} \times (p-1)^2}.$$

Say $\mathsf{Robins}(n)$ holds provided

$$G(n) < e^{\gamma}$$

where the constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The importance of this property is:

Proposition 1.3. $\mathsf{Robins}(n)$ *holds for all natural numbers $n > 5040$ if and only if the Riemann Hypothesis is true [3].*

The Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x [4]. We state the following property about this function:

Proposition 1.4. *For every $x \geq 19035709163$ [5]:*

$$\theta(x) > (1 - \frac{0.15}{\log^3 x}) \times x.$$

In mathematics, $\Psi = n \times \prod_{q|n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function. Say $\mathsf{Dedekinds}(q_n)$ holds provided

$$\frac{\pi^2}{6} \times \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > e^{\gamma} \times \log \theta(q_n)$$

where q_n is the nth prime number. The importance of this inequality is:

Proposition 1.5. $\mathsf{Dedekinds}(q_n)$ *holds for all prime numbers $q_n > 3$ if and only if the Riemann Hypothesis is true [6].*

Let $q_1 = 2, q_2 = 3, \ldots, q_k$ denote the first k consecutive primes, then an integer of the form $\prod_{i=1}^{k} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$ is called an Hardy-Ramanujan integer [7]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

Proposition 1.6. *If n is superabundant, then n is an Hardy-Ramanujan integer [8]. Let n be a superabundant number, then $p \parallel n$ where p is the largest prime factor of n except when $n \in \{4, 36\}$ [8]. For large enough superabundant number n, we have that $q^{a_q} < 2^{a_1}$ for $q > 11$ where $q^{a_q} \parallel n$ and $2^{a_1} \parallel n$ [8]. For large enough superabundant number n, we obtain that $\log n < (1 + \frac{0.5}{\log p}) \times p$ where p is the largest prime factor of n [4]. Let n be a superabundant* *number, then* $f(n) > (1 - \varepsilon(p)) \times \prod_{q|n} \frac{q}{q-1}$ *where* $\varepsilon(p) = \frac{1}{\log p} \times (1 + \frac{1.5}{\log p})$ *and* p *is the largest prime factor of* n *[4]. Let* n *be a superabundant number such that* $q^{a_q} \parallel n$ *and* $2 \leq q \leq p$*, then*

$$\left\lfloor \frac{\log p}{\log q} \right\rfloor \leq a_q$$

where p *is the largest prime factor of* n *and* $\lfloor \ldots \rfloor$ *is the floor function [8]. If* n *is superabundant, then*

$$p \sim \log n$$

where p *is the largest prime factor of* n *[8]. There are infinitely many superabundant numbers, since the number of superabundant numbers less than* x *exceeds:*

$$\frac{c \times \log x \times \log \log x}{(\log \log \log x)^2}$$

for some constant $c > 0$ *[8].*

In addition, we will use these properties:

Proposition 1.7. *[6], [7]. For* $n \geq 2$*:*

$$\prod_{q > q_n} \frac{q^2}{q^2 - 1} \leq e^{\frac{2}{q_n}}.$$

Proposition 1.8. *It is known that [9]:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6}.$$

In number theory, a perfect number is a positive integer n such that $f(n) = 2$. Euclid proved that every even perfect number is of the form $2^{s-1} \times (2^s - 1)$ whenever $2^s - 1$ is prime. It is unknown whether any odd perfect numbers exist, though various results have been obtained:

Proposition 1.9. *Any odd perfect number* N *must satisfy the following conditions:* $N > 10^{1500}$ *and the largest prime factor of* N *is greater than* 10^8 *[10], [11].*

Now, we state the following conjecture:

Conjecture 1.10. *We assume that the Riemann Hypothesis is true and the inequality*

$$\frac{\pi^2}{\beta} \times \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > e^{\gamma} \times \log \theta(q_n)$$

is satisfied for infinitely many prime numbers q_n *and* $\beta \geq 6.0008$.

Under the assumption that the Conjecture 1.10, we prove that there is not any odd perfect number at all.

2. The Main Insight

Theorem 2.1. *For any natural number N, there are always infinitely many large enough superabundant numbers n such that n is a multiple of N.*

Proof. Suppose that p is the largest prime factor of the superabundant number n. We will use the Proposition 1.6 in this proof. We have that there are infinitely many superabundant numbers n. Moreover, we know that

$$p \sim \log n$$

which means that p goes to infinity as long as n goes to infinity. In addition, every prime number q between 2 and p divides n due to n must be an Hardy-Ramanujan integer when n is superabundant. Furthermore, for every prime power $q^{a_q} \parallel n$ and $2 \leq q \leq p$, we obtain that

$$\left\lfloor \frac{\log p}{\log q} \right\rfloor \leq a_q$$

which implies that

$$p \leq q^{a_q+1}.$$

That would mean that a_q goes to infinity as long as p goes to infinity. In this way, for every prime power q^{a_q}, there are always infinitely many large enough superabundant numbers n such that n is a multiple of q^{a_q}. Since every natural number N has a prime factorization within prime powers, then the proof is done. □

3. The Main Theorem

Theorem 3.1. *Under the assumption that the Conjecture 1.10, we prove that there is not any odd perfect number at all.*

Proof. Suppose that N is the smallest odd perfect number, then we will show its existence implies that the Conjecture 1.10 is false. There are always infinitely many large enough superabundant numbers n such that n is a multiple of N because of the Theorem 2.1. We could take the largest prime factor p of n as $p > 10^{10000}$ according to the Proposition 1.9 and the Theorem 2.1. We would have

$$f(n) \leq f(N) \times f(\frac{n}{N})$$

according to the Proposition 1.2. That is the same as

$$f(n) \leq 2 \times f(\frac{n}{N})$$

since $f(N) = 2$, because N is a perfect number. Hence,

$$\begin{aligned}\frac{f(n)}{2} &= \frac{(2 - \frac{1}{2^{a_1}}) \times f(\frac{n}{2^{a_1}})}{2} \\ &= f(\frac{n}{2^{a_1}}) \times \frac{(2 - \frac{1}{2^{a_1}})}{2} \\ &= f(\frac{n}{2^{a_1}}) \times \frac{2^{a_1+1} - 1}{2^{a_1+1}}\end{aligned}$$

when $2^{a_1} \parallel n$ due to the Proposition 1.2. In this way, we have

$$\frac{f(\frac{n}{2^{a_1}})}{f(\frac{n}{N})} \leq \frac{2^{a_1+1}}{2^{a_1+1}-1}.$$

However, we know that $p < 2^{a_1}$ because of $p > 10^{10000} > 11$ and the Propositions 1.6 and 1.9. Consequently,

$$\frac{2^{a_1+1}}{2^{a_1+1}-1} \leq \frac{2\times p}{2\times p-1}$$

since $\frac{x}{x-1}$ decreases when $x \geq 2$ increases. In addition, we know that

$$\frac{2\times p}{2\times p-1} \leq f(p)$$

where we know that $f(p) = \frac{p+1}{p}$ from the Proposition 1.2. Certainly,

$$\begin{aligned} 2\times p^2 &\leq (p+1)\times(2\times p-1) \\ &= 2\times p^2 + 2\times p - p - 1 \\ &= 2\times p^2 + p - 1 \end{aligned}$$

where this inequality is satisfied for every prime number p. So,

$$\frac{f(\frac{n}{2^{a_1}})}{f(\frac{n}{N})} \leq f(p)$$

where we know that $p \parallel n$ from the Proposition 1.6. Using the Conjecture 1.10, we have that

$$\begin{aligned} e^{\gamma} &> G(n) \\ &= \frac{f(\frac{n}{p})\times f(p)}{\log\log n} \\ &\geq \frac{f(\frac{n}{p})\times f(\frac{n}{2^{a_1}})}{f(\frac{n}{N})\times \log\log n} \end{aligned}$$

since $f(\ldots)$ is multiplicative and as a consequence of the Proposition 1.3. This is equivalent to

$$\frac{f(\frac{n}{p})}{f(\frac{n}{N})} < \frac{e^{\gamma}}{f(\frac{n}{2^{a_1}})} \times \log\log n.$$

Under the assumption that the Conjecture 1.10, we deduce that:

$$\frac{\pi^2}{6.0008} \times \prod_{q\leq p}\left(1+\frac{1}{q}\right) > e^{\gamma}\times \log\theta(p)$$

which is the same as

$$\frac{\pi^2}{8} \times \prod_{q\leq p}\left(1+\frac{1}{q}\right) > e^{\gamma}\times \log((\theta(p))^{0.7501}).$$

From the Propositions 1.1 and 1.6, we know that

$$f(\frac{n}{2^{a_1}}) = \left(\prod_{i=2}^{k} \frac{q_i}{q_i - 1}\right) \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right)$$

where $q_k = p$ and $q_1 = 2$. We know that

$$\frac{q_i}{q_i - 1} = \left(1 + \frac{1}{q_i}\right) \times \frac{q_i^2}{q_i^2 - 1}.$$

Using the previous inequality and the Conjecture 1.10, we obtain that

$$\begin{aligned} e^{\gamma} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \times \log((\theta(p))^{0.7501}) &< \frac{\pi^2}{8} \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &= f(\frac{n}{2^{a_1}}) \times \frac{3}{2} \times \prod_{q > p} \frac{q^2}{q^2 - 1} \\ &\leq f(\frac{n}{2^{a_1}}) \times \frac{3}{2} \times e^{\frac{2}{p}} \end{aligned}$$

according to the Proposition 1.7. Taking into account that $p > 10^{10000} > 3$ and n is superabundant:

$$\frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.7501})} > \frac{e^{\gamma}}{f(\frac{n}{2^{a_1}})} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

We use the previous inequality to show that

$$\frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) < \frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.7501})} \times \log\log n.$$

For large enough superabundant number n and $p > 10^{10000}$, then

$$\frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.7501})} \times \log\log n \leq \frac{\frac{3}{2} \times e^{\frac{2}{10^{10000}}}}{\log\left(((1 - \frac{0.15}{\log^3 10^{10000}}) \times 10^{10000})^{0.7501}\right)} \times \log\left((1 + \frac{0.5}{\log 10^{10000}}) \times 10^{10000}\right)$$

because of the Propositions 1.4 and 1.6. We obtain that

$$\frac{\frac{3}{2} \times e^{\frac{2}{10^{10000}}}}{\log\left(((1 - \frac{0.15}{\log^3 10^{10000}}) \times 10^{10000})^{0.7501}\right)} \times \log\left((1 + \frac{0.5}{\log 10^{10000}}) \times 10^{10000}\right) < 1.999733371.$$

Thus,

$$\frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) < 1.999733371.$$

For every prime p_j that divides N such that $p_j^{a_j} \parallel N$ and $p_j^{a_j+b_j} \parallel n$ for a_j, b_j two natural numbers, we have that

$$f(p_j^{a_j+b_j}) - f(p_j^{a_j}) \times f(p_j^{b_j}) = -\frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{p_j^{a_j+b_j-1} \times (p_j - 1)^2}$$

in the Proposition 1.2. This is equal to

$$\frac{f(p_j^{a_j+b_j})}{f(p_j^{b_j})} = f(p_j^{a_j}) - \frac{(p_j^{a_j}-1)\times(p_j^{b_j}-1)}{f(p_j^{b_j})\times p_j^{a_j+b_j-1}\times(p_j-1)^2}.$$

Hence,

$$\begin{aligned}
\frac{f(\frac{n}{p})}{f(\frac{n}{N})}\times\prod_{i=2}^{k}\left(1-\frac{1}{q_i^{a_i+1}}\right) &= \prod_j\left(\frac{f(p_j^{a_j+b_j})}{f(p_j^{b_j})}\right)\times\prod_{i=2}^{k}\left(1-\frac{1}{q_i^{a_i+1}}\right)\times\frac{1}{f(p)}\\
&= \prod_j\left(f(p_j^{a_j}) - \frac{(p_j^{a_j}-1)\times(p_j^{b_j}-1)}{f(p_j^{b_j})\times p_j^{a_j+b_j-1}\times(p_j-1)^2}\right)\times\prod_{i=2}^{k}\left(1-\frac{1}{q_i^{a_i+1}}\right)\times\frac{1}{f(p)}\\
&> 1.9999\times\prod_{i=2}^{k}\left(1-\frac{1}{q_i^{a_i+1}}\right)\times\left(1-\frac{1}{p+1}\right)\\
&> 1.9999\times\left(1-\frac{1}{\log p}\times(1+\frac{1.5}{\log p})\right)\times\frac{(1-\frac{1}{p+1})}{(1-\frac{1}{2^{a_1+1}})}\\
&> 1.9999\times\left(1-\frac{1}{\log p}\times(1+\frac{1.5}{\log p})\right)\times\left(1-\frac{1}{p+1}\right)\\
&> 1.9999\times\left(1-\frac{1}{\log 10^{10000}}\times(1+\frac{1.5}{\log 10^{10000}})\right)\times\left(1-\frac{1}{10^{10000}+1}\right)\\
&> 1.999733371
\end{aligned}$$

using the Propositions 1.6 and 1.1 since we know that the expression

$$\frac{(p_j^{a_j}-1)\times(p_j^{b_j}-1)}{f(p_j^{b_j})\times p_j^{a_j+b_j-1}\times(p_j-1)^2}$$

tends to 0 as b_j tends to infinity for every odd prime p_j where

$$\begin{aligned}
\prod_j\left(f(p_j^{a_j}) - \frac{(p_j^{a_j}-1)\times(p_j^{b_j}-1)}{f(p_j^{b_j})\times p_j^{a_j+b_j-1}\times(p_j-1)^2}\right) &\approx \prod_j\left(f(p_j^{a_j})\right)\\
&= f(N)\\
&= 2.
\end{aligned}$$

Certainly, the fraction $\frac{f(\frac{n}{p})}{f(\frac{n}{N})}$ gets closer to 2 as long as we take n bigger and bigger. In addition, we note that

$$\begin{aligned}
\left(1-\frac{1}{\log p}\times(1+\frac{1.5}{\log p})\right) &< \prod_{i=1}^{k}\left(1-\frac{1}{q_i^{a_i+1}}\right)\\
&= \prod_{i=2}^{k}\left(1-\frac{1}{q_i^{a_i+1}}\right)\times(1-\frac{1}{2^{a_1+1}})
\end{aligned}$$

after taking into account the Proposition 1.6. However,

$$1.999733371 < \frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) < 1.999733371$$

is a contradiction. By contraposition, the number N does not exist under the assumption that the Conjecture 1.10. The smallest counterexample N must comply that $N > 10^{1500}$ and therefore, we will always be capable of obtaining a large enough superabundant number n that is multiple of N. Note that, this proof fails for even perfect numbers or for some other odd numbers N such that $f(N) > 2$ (precisely when we may consider a large enough superabundant number n). □

Acknowledgments

The author would like to thank Patrick Solé for helpful comments and his mother and friends Sonia and Arelis for their support. We intend to publish this paper with Patrick Solé if he finally agrees to be co-author of a full, final and public article. This incomplete version only meant to reveal progress in case something unfortunate and unwanted happens.

References

[1] A. Hertlein, Robin's Inequality for New Families of Integers, Integers 18, (2018).

[2] R. Vojak, On numbers satisfying Robin's inequality, properties of the next counterexample and improved specific bounds, arXiv preprint arXiv:2005.09307, (2020).

[3] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann, J. Math. pures appl 63 (2) (1984) 187–213.

[4] S. Nazardonyavi, S. Yakubovich, Superabundant numbers, their subsequences and the Riemann hypothesis, arXiv preprint arXiv:1211.2147, (2012).

[5] C. Axler, New Estimates for Some Functions Defined Over Primes, Integers 18, (2018).

[6] P. Solé, M. Planat, Extreme values of the Dedekind ψ function, Journal of Combinatorics and Number Theory 3 (1) (2011) 33–38.

[7] Y. Choie, N. Lichiardopol, P. Moree, P. Solé, On Robin's criterion for the Riemann hypothesis, Journal de Théorie des Nombres de Bordeaux 19 (2) (2007) 357–372. doi:10.5802/jtnb.591.

[8] L. Alaoglu, P. Erdős, On highly composite and similar numbers, Transactions of the American Mathematical Society 56 (3) (1944) 448–469. doi:10.2307/1990319.

[9] H. M. Edwards, Riemann's Zeta Function, Dover Publications, 2001.

[10] P. Ochem, M. Rao, Odd perfect numbers are greater than 10^{1500}, Mathematics of Computation 81 (279) (2012) 1869–1877. doi:10.1090/S0025-5718-2012-02563-4.

[11] T. Goto, Y. Ohno, Odd perfect numbers have a prime factor exceeding 10^8, Mathematics of Computation 77 (263) (2008) 1859–1868.