Infinite Servers Queues and Diffusion Stochastic Processes in Equilibrium and Maintenance Costs of Pensions Funds Study

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Abstract

In this study is depicted a representation of a pensions fund through a stochastic network with two infinite servers 'nodes. With this representation it is allowed to deduce an equilibrium condition of the system with basis on the identity of the random rates expected values, for which the contributions arrive to the fund and the pensions are paid by the fund. Then, to address situations of imbalance, the generic case of a pensions fund that it is not sufficiently auto financed, and it is thoroughly maintained with an external financing effort is considered in this chapter. To represent the unrestricted reserves value process of this kind of fund, a time homogeneous diffusion stochastic process with finite expected time to ruin is proposed. Then it is projected a financial tool that regenerates the diffusion at some level with positive value every time the diffusion hits a barrier placed at the origin. So, the financing effort can be modeled as a renewal-reward process if the regeneration level is preserved constant. The perpetual maintenance cost expected values and the finite time maintenance cost evaluations are studied. An application of this approach when the unrestricted reserves value process behaves as a generalized Brownian motion process is presented.

Keywords: pensions fund, system equilibrium, Poisson process, Wald's equation diffusion process, first passage times, perpetuity, renewal equation.

MSC: 60G07

1.Introduction

Pension funds represent savings collected throughout people's working life. Pension funds that support personal pension plans are intended to be autonomous. They represent the highest level of protection to the beneficiary from bankruptcy of the sponsor, especially when the custodian is involved. Non-autonomous pension funds are not legally separated from the plan sponsor but are kept on their balance sheet. In this case, there is the lowest protection level to the beneficiary from bankruptcy of the sponsor, since the sponsor can use pension's assets to fund its business, see [1].

First, we present a study on the sustainability of a pensions fund see for instance [2,3, and 4] in the field of queuing theory. Two infinite servers' queues are considered: one with the contributors to the fund, which service time is the time during which they contribute to the fund; the other with the pensioners which service time is the time during which they receive the pension. In both queues, there is no distinction between a customer and its server in technical sense.

The most important consequence of the previous study is that for the fund to be balanced, the average pension must be equal to the average contribution. However, due to the demographic imbalance that exists in contemporary societies, with the number of contributors successively decreasing while the number of retirees increases inexorably, this meant that, if the funds were to be balanced autonomously, workers' contributions would have to assume unbearable values.

The usual way to address this problem is to inject capital into these resources through transfers from the Public Budget whenever necessary. Therefore, in a disorganized way and, in general, in unforeseen situations that may coincide with moments of great financial difficulties.

So, the objective of the subsequent study is to try to make these situations more predictable, both in relation to the moment of occurrence and the amount needed, so that the protection of these funds occurs as smoothly as possible.

The financial problem of asset-liability management scheme of a pensions fund requires a management program that demands a set of decisions. In particular, the amounts and the instants at which it is necessary to inject money in the fund to keep it sustainable. Sponsors are obviously interested in an appropriate management of the risk for their pension funds. Well and balanced funded pension funds result essential in this process of funds management.

Through this chapter we also will develop a mathematical tool that allows predicting, in a probabilistic mode, the appropriate moments to carry out these money injections and the respective amounts.

This issue is particularly relevant since we know that pension funds are continuously exposed to the market's situation. And the recent financial crises and turbulent stock markets circumstances made the problem of pension funds management receive attention. Many pensions' funds suffered dramatic losses, and this is a problematic issue that managers want to overcome the best they can. So, managerial tools allow a better decision-making.

The protection cost present value expectation for a non-autonomous pensions' fund is considered in this work. Two contexts are considered:

- The protection effort is perpetual,
- The protection effort happens for a finite period.

It is admitted that the unrestricted fund reserves behavior may be modeled as a time homogeneous diffusion process. Then a regeneration scheme of the diffusion to include the effect of an external financing effort is used.

In this part, this chapter is an updated and enlarged version of [5], where was mainly considered the diffusion process.

In [6] a similar work is presented. A Brownian motion process conditioned by a reflection scheme was considered. With less constraints, but in different conditions, exact solutions were then obtained for both problems.

The work presented in [7] , on asset-liability management aspects, also motivated the use of the Brownian motion application example in that domain.

So, in this chapter we extend the results presented in [5], better specifying the diffusion process mathematical details, and deeply exploring the Brownian motion process situation

Other works on this subject are [8, and 9] both dealing with the diffusion process case. The works [10, 11, and 12], deal with other financial problems, slightly different from the presently considered here, but relevant to their understanding and framing.

2.The Fund Equilibrium

Begin assuming two nodes, service centers, *A* and *B,* both with infinite servers. The traffic through arches *a* to *e* is as it is schematized in Figure 1.

$$\begin{array}{ccccccc} & & \uparrow & & | & & \\ & & c & & d & & \\ & & | & & \downarrow & & \\ -a\rightarrow & & \blacksquare & -b\rightarrow & \blacksquare & & -e\rightarrow \\ & & A & & B & & \end{array}$$

Figure 2.1. Traffic in the stochastic network

The users arrive to node *A* by arch *a* at rate λ_A. And the sojourn time at this node is a positive random variable with distribution function (*d.f.*) $G_A(.)$ and finite mean α_A. After node *A* the users go to node *B* through *b* with probability *p.* Or just abandon the system through arch *c* with probability $1-p$. There may also be users coming directly from outside through *d* at rate λ_B to node *B*. The sojourn time at this node is a positive random variable with *d.f.* $G_B(.)$ and finite mean α_B. The system is abandoned by these users through arch *e.*

This system is suggested as a representation of a pensions fund. Therefore, at node *A* are individuals whose service consists in paying, during their working time, their contributions to the fund. The pensioners are at node *B*, whose service represents their pensions payment by the fund. In this representation it is also reflected the functions of the common social security funds and that is why it accepts the access of pensioners that have not formerly participated, at node *A*, in the building of the fund.

By considering this representation, in this approach, the objective is to obtain results about the transient behavior of the system from the point of view of its equilibrium and autonomy.

So, name $N_A(t)$ and $N_B(t)$ be the random variables (*r.v.*) that represent the number of individuals by time *t* at nodes *A* and *B,* respectively. Also define the sets of random variables, *i.i.d*: $X_{A_1}(t), X_{A_2}(t), X_{A_3}(t), \dots, \left(X_{B_1}(t), X_{B_2}(t), X_{B_3}(t), \dots\right)$ which designate the unitary contributions, pensions by time *t*, with mean $m_A(t)$ and $m_B(t)$[1].

Assuming that the system is in equilibrium when the expected values of the rates at which the contributions are being received and the pensions are being paid by the fund are identical:

$$E\left[\sum_{i=1}^{N_A(t)} X_{A_i}(t)\right] = E\left[\sum_{j=1}^{N_B(t)} X_{B_j}(t)\right] \qquad (2.1).$$

That is, by Wald's equation:

$$m_A(t)E[N_A(t)] = m_B(t)E[N_B(t)] \qquad (2.2).$$

So, at each instant the mean value of the unitary pension should be proportional to the mean value of the unitary contribution, with the ratio between the averages of the numbers of contributors and pensioners as proportionality factor. Being $t = 0$ the origin time, its solution corresponds, for $t > 0$, to the following pairs:

$$\left(m_A(t); m_B(t)\right) = \left(m_A(t); \frac{m_A(t)E[N_A(t)]}{E[N_B(t)]}\right) \qquad (2.3),$$

where $m_A(t)$ is independent of the equilibrium.

If the mean value of the unitary pension is initially 1, and grows continuously with an interest rate *r*,

$$\begin{gathered} m_B(t) = e^{rt} \\ m_A(t) = e^{rt}\left(E[N_B(t)]/E[N_A(t)]\right) \end{gathered} \qquad (2.4)$$

It is elementary, after (2.1)), that

$$E[N_A(t)] < E[N_B(t)] \Rightarrow m_A(t) > m_B(t) \qquad (2.5).$$

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[1] Under the term unitary contribution/pension, it is meant the amount of money that one individual pays/receives, by unit of time.

Thus, being the system in equilibrium, the mean value of the unitary pension is smaller than the mean value of the unitary contribution whenever the mean number of pensioners at B is bigger than the mean number of contributors at A.

3.Poisson Arrivals Situation

When considering the arrivals from outside at nodes A and B according to a Poisson process, with rates λ_A and λ_B, respectively, the system may be seen as a two nodes network where the first node is a $M|G|\infty$ queue and second $M_t|G|\infty$ queue, see for instance [13]. So, $N_A(t)$ is exponentially distributed with parameter, see also [13],

$$\lambda_A \int_0^t \left(1 - G_A(v)\right) dv \quad (3.1).$$

The output of the first node is a non-homogeneous Poisson process with intensity function $\lambda_A G_A(t)$ and, consequently, the global arrivals rate at node B is $p\lambda_A G_A(t) + \lambda_B$. Under this conditions $N_B(t)$ is quasi-exponentially distributed with parameter, see [13] :

$$\int_0^t (p\lambda_A G_A(v) + \lambda_B)\left(1 - G_B(t - v)\right) dv \qquad (3.2).$$

And (2.2) is now written as:

$$m_A(t)\lambda_A \int_0^t \left(1 - G_A(v)\right) dv = m_B(t) \int_0^t (p\lambda_A G_A(v) + \lambda_B)\left(1 - G_B(t - v)\right) dv \qquad (3.3).$$

Now, versions of these results some concrete examples of service times distributions follow:

Uniformly Distributed Service Times

Assuming the service times are uniformly distributed, supposing that $\alpha_B < \alpha_A$, it is obtained for (2.2) in $0 \le t < 2\alpha_A + 2\alpha_B$:

i) $m_A(t)\lambda_A \left(t - \frac{t^2}{4\alpha_A}\right) = m_B(t)\lambda_B \left(t - \frac{t^2}{4\alpha_B}\right) + m_B(t)p\lambda_A \left(\frac{t^2}{4\alpha_A} - \frac{t^3}{24\alpha_A\alpha_B}\right),\ 0 \le \frac{t}{2} < \alpha_B$
(3.4),

ii) $m_A(t)\lambda_A \left(t - \frac{t^2}{4\alpha_A}\right) = m_B(t)\lambda_B\alpha_B + m_B(t)p\lambda_A \left(-\frac{\alpha_B^2}{3\alpha_A} - \frac{t\alpha_B}{2\alpha_A}\right),\ \alpha_B \le \frac{t}{2} < \alpha_A$
(3.5),

$iii)\ \ m_A(t)\lambda_A\alpha_A = m_B(t)\lambda_B\alpha_B + m_B(t)p\lambda_A\left(-\alpha_A - \frac{\alpha_A^2}{12\alpha_B} + t - \frac{(t-\alpha_A)^2}{4\alpha_B} + \frac{(t-2\alpha_B)^3}{24\alpha_A\alpha_B}\right),\ \alpha_A \le \frac{t}{2} < \alpha_A + \alpha_B$ (3.6).

Exponentially Distributed Service Times

Now, considering that the service times are exponentially distributed the equilibrium distribution is given by:

$i)\ \ m_A(t)\lambda_A\alpha_A\left(1 - e^{-\frac{t}{\alpha_A}}\right) = m_B(t)(p\lambda_A + \lambda_B)\alpha_B\left(1 - e^{-\frac{t}{\alpha_B}}\right) - m_B(t)\frac{\rho\lambda_A\alpha_A\alpha_B}{\alpha_A-\alpha_B}\left(e^{-\frac{t}{\alpha_A}} - e^{-\frac{t}{\alpha_B}}\right),\ \alpha_A \neq \alpha_B$ (3.7),

$ii)\ \ m_A(t)\lambda_A\alpha_A\left(1 - e^{-\frac{t}{\alpha_A}}\right) = m_B(t)\ (p\lambda_A + \lambda_B)\alpha_A\left(1 - e^{-\frac{t}{\alpha_A}}\right) - m_B(t)p\lambda_A t e^{-\frac{t}{\alpha_A}},\ \ \alpha_A = \alpha_B$ (3.8).

Service Times with a Particular Distribution Function

To solve (2.2) in the way presented above becomes quite difficult with other standard distributions for the service times. So now it will be presented a collection of *d.f.*'s, see [14, and 15], for the service times given by

$G_i(v) = 1 - \frac{(1-e^{-\rho_i})(\gamma_i+\beta_i)}{\gamma_i e^{-\rho_i}\left(e^{(\gamma_i+\beta_i)v}-1\right)+\gamma_i}, v \ge 0, \gamma_i > 0, \rho_i > 0, -\gamma_i \le \beta_i \le \frac{\gamma_i}{e^{-\rho_i}-1}, i = A, B$ (3.9).

The mean distribution is $\alpha_i = \rho_i/\gamma_i$. In this case (2.2) becomes:

$$m_A(t)\frac{\lambda_A}{\gamma_A} ln\frac{e^{(\gamma_A+\beta_A)t}}{e^{-\rho_A}(e^{(\gamma_A+\beta_A)t} - 1) + 1} = m_B(t)\frac{p\lambda_A + \lambda_B}{\gamma_B} ln\frac{e^{(\gamma_B+\beta_B)t}}{e^{-\rho_B}(e^{(\gamma_B+\beta_B)t} - 1) + 1} - m_B(t)p\lambda_A I(t)$$

where $I(t) = \int_0^t \frac{(1-e^{-\rho_A})(\gamma_A+\beta_A)}{\gamma_A e^{-\rho_A}\left(e^{(\gamma_A+\beta_A)v}-1\right)+\gamma_A} \times \frac{(1-e^{-\rho_B})(\gamma_B+\beta_B)}{\gamma_B e^{-\rho_B}\left(e^{(\gamma_B+\beta_B)(t-v)}-1\right)+\gamma_B} dv,$ (3.10).

$I(t)$ is non-negative and not bigger than

$$\frac{(\gamma_A + \beta_A)(\gamma_A + \beta_A)^t}{\gamma_A + \gamma_B}, (3.11)$$

Approximations

The solutions for (2.2) appear to be significantly more complex in circumstances different from those that have been mentioned. For instance, if the service times follow a Lognormal, Gama or Weibull distributions. In some cases, only the numerical solution can eventually be stained.

For appropriate values of t, the following approximations concerning the equilibrium conditions are suggested:

$$\frac{m_B(t)}{m_A(t)} \cong \frac{\lambda_A \alpha_A}{(p\lambda_A + \lambda_B)\alpha_B} \qquad (3.12);$$

$$\frac{m_B(t)}{m_A(t)} \cong \frac{\lambda_A}{\lambda_B} \qquad (3.13).$$

Equation (3.13) seems reasonable for values of t big enough and (3.13) is preferrable for t close to zero. For details see [4].

Observations:

-Some values of the parameters p and λ_B have a special influence in the system behaviour. One may consider the suppression of the arch b when $p = 0$, of the arch c when $p = 1$ or of the arch d for $\lambda_B = 0$. Under those circumstances the traffic in those arches can be neglected,

-It may be admitted that the ratio $m_B(t)/m_A(t)$ remains constant. This corresponds to the assumption that all the users of the system face identical conditions of effort and benefit, independently of the moment they join the system. Equation (3.12) supplies a natural candidate for the value of that constant: $\lambda_A \alpha_A/(p\lambda_A + \lambda_B)\alpha_{B.}$ In such situation (3.3) should include an "excess" functions $h(t)$:

$$h(t) = m_B(t)\frac{\lambda_A \alpha_A}{(p\lambda_A+\lambda_B)\alpha_B}\int_0^t (p\lambda_A G_A(v) + \lambda_B)\big(1 - G_B(t-v)\big)\,dv - m_A(t)\lambda_A\int_0^t \big(1 - G_A(v)\big)dv \qquad (3.14).$$

The function $h(t)$ is also interpreted in the sense of the expected value of a random variable depending on t. This approach can be generalized in a natural way to some other predefined function $m_B(t)/m_A(t)$,

-Assuming that the system is initially empty appears to be a strong restriction of the analysis performed. When someone meets the system already in operation and does not know when it did start, the results that have been mentioned seem to have a lesser utility. In such case, there re-evaluation or finding an estimation procedure for the initial time are determinant for practical purposes.

- Evidently, if contributors successively decrease while the number of retirees increases, this means that, if the funds were to be balanced autonomously, workers' contributions would have to assume unbearable values. The usual way to address this problem is to inject capital into these resources through transfers from the Public Budget whenever necessary. Next, we will present a model to study this procedure.

4. Pensions Fund Reserves Behavior Stochastic Model

Think $X(t), t \geq 0$ the reserves value process of a pensions fund given by an initial reserve amount $a, a > 0$ added to the difference between the total amount of contributions received and the total amount of pensions paid both up to time t. Assume $X(t)$ is a time homogeneous diffusion process, with $X(0) = a$, defined by drift $\mu(x)$, and diffusion coefficient $\sigma^2(x)$.

Call S_a the first passage time of $X(t)$ by 0, coming from a. The funds to be considered in this work are non-autonomous funds. So

$$E[S_a] < \infty, \text{for any } a > 0 \qquad (4.1),$$

That is: funds where the pensions paid consume in finite expected time any initial positive reserve and the contributions received. Then other financing resources are needed to fund survival.

The condition (4.1) may be fulfilled for a specific diffusion process using criteria based on the drift and diffusion coefficients. In this context, the work presented in $[16, \text{pg}. 418 - 422]$, will be followed.

So, accept $P(S_a < \infty) = 1$ if the diffusion scale function is $q(x) = \int_{x_0}^{x} e^{-\int_{x_0}^{z} \frac{2\mu(y)}{\sigma^2(y)} dy} dz$, where x_0 is a diffusion state space fixed arbitrary point, fulfilling $q(\infty) = \infty$. Then the condition (4.1) is equivalent to $p(\infty) < \infty$, where $p(x) = \int_{x_0}^{x} \frac{2}{\sigma^2(z)} e^{\int_{x_0}^{z} \frac{2\mu(y)}{\sigma^2(y)} dy} dz$, is the diffusion speed function.

It is admitted that whenever the exhaustion of the reserves happens an external source place instantaneously an amount $\theta, \theta > 0$ of money in the fund so that it may keep on performing its function.

The reserves value process conditioned by this financing scheme is denoted by the modification $\check{X}(t)$ of $X(t)$ that restarts at the level θ whenever it hits 0. As $X(t)$ was defined as a time homogeneous diffusion, $\check{X}(t)$ is a regenerative process. Call $T_1, T_2, T_3, \ldots$ the sequence of random variables where T_n denotes the n^{th} $\check{X}(t)$ passage time by 0. It is obvious that the sequence of time intervals between these hitting times $D_1 = T_1, D_2 = T_2 - T_1, D_3 = T_3 - T_2, \ldots$ is a sequence of independent random variables where D_1 has the same probability distribution as S_a and $D_2, D_3, \ldots$ the same probability distribution as S_θ.

First Passage Times Laplace Transforms

Call $f_a(s)$ the probability density function of S_a (related to D_1). The corresponding probability distribution function is denoted by $F_a(s)$. The Laplace transform of S_a is denoted $\varphi_a(\lambda)$.

Consequently, the density, distribution and transform of S_θ (related to $D_2, D_3, \ldots$) will be denoted by $f_\theta(s), F_\theta(s)$ and $\varphi_\theta(\lambda)$, respectively.

The transform $\varphi_a(\lambda)$ satisfies the second order differential equation

$$\begin{aligned} &\tfrac{1}{2}\sigma^2(a)u_\lambda''(a) + \mu(a)u_\lambda'(a) = \lambda u_\lambda(a), \\ &u_\lambda(a) = \varphi_a(\lambda), u_\lambda(0) = 1, u_\lambda(\infty) = 0 \end{aligned} \qquad (4.2),$$

See $[16, \text{pg. } 243; 17, \text{pg.} 478; 18, \text{pg. } 89]$.

Perpetual Maintenance Cost Present Value

Consider the perpetual maintenance cost present value of the pension's fund given by the random variable$V(r, a, \theta) = \sum_{n=1}^{\infty} \theta e^{-rT_n}$, r>0, where r represents the appropriate discount rate. Note that $V(r, a, \theta)$ is a random perpetuity. What matters is its expected value which is simple to calculate through Laplace transforms. Since the T_n Laplace transform is$E\left[e^{-\lambda T_n}\right] = \varphi_a(\lambda)\varphi_\theta^{n-1}(\lambda)$,

$$v_r(a, \theta) = E[V(r, a, \theta)] = \frac{\theta\varphi_a(r)}{1 - \varphi_\theta(r)} \qquad (4.3).$$

It is relevant to note that

$$\lim_{\theta \to 0} v_r(a, \theta) = \frac{u_r(a)}{-u_r'(0)} \qquad (4.4) .$$

Finite Time Period Maintenance Cost Present Value

Define the renewal process $N(t)$ as $N(t) = \sup\{n: T_n \leq t\}$, generated by the extended sequence $T_0 = 0, T_1, T_2, \dots$. The present value of the pensions fund maintenance cost up to time t is represented by the stochastic process

$$W(t; r, a, \theta) = \sum_{n=1}^{N(t)} \theta e^{-rT_n}, \ W(t; r, a, \theta) = 0 \text{ if } N(t) = 0.$$

To calculate the expected value function of the process evaluation: $w_r(t; a, \theta) = E[W(t; r, a, \theta)]$, begin noting that it may be expressed as a numerical series. Indeed, evaluating the expected value function conditioned by $N(t) = n$, it is obtained $E[W(t; r, a, \theta)|N(t) = n] = \theta\varphi_a(r)\frac{1-\varphi_\theta^n(r)}{1-\varphi_\theta(r)}$. Repeating the expectation:

$$w_r(t; a, \theta) = E[E[W(t; r, a, \theta)]|N(t)] = \theta\varphi_a(r)\frac{1-\gamma(t,\varphi_\theta(r))}{1-\varphi_\theta(r)} \qquad (4.5).$$

Here $\gamma(t, \xi)$ is the probability generating function of $N(t)$.

Denote now the T_n probability distribution function by $G_n(s)$ and assume $G_0(s) = 1$, for $s \geq 0$. Recalling that $P(N(t) = n) = G_n(t) - G_{n+1}(t)$, the above-mentioned probability generating function is

$$\gamma(t, \xi) = \sum_{n=0}^{\infty} \xi^n \, P(N(t) = n) = 1 - (1 - \xi) \sum_{n=1}^{\infty} \xi^{n-1} \, G_n(t) \qquad (4.6).$$

Substituting (4.6) in (4.5), $w_r(t; a, \theta)$ is expressed in the form of the series:

$$w_r(t; a, \theta) = \theta\varphi_a(r) \sum_{n=1}^{\infty} \varphi_\theta^{n-1}(r) \, G_n(t) \qquad (4.7).$$

Call the $w_r(t;a,\theta)$ ordinary Laplace transform $\psi(\lambda)$. The probability distribution function $G_n(s)$, of T_n, ordinary Laplace transform is given $\varphi_a(\lambda)\frac{\varphi_\theta^{n-1}(\lambda)}{\lambda}$ and performing the Laplace transforms in both sides of (4.7) it is obtained $\psi(\lambda) = \frac{\theta\varphi_a(r)\varphi_a(\lambda)}{\lambda(1-\varphi_\theta(r)\varphi_\theta(\lambda))}$ or

$$\psi(\lambda) = \theta\varphi_a(r)\frac{\varphi_a(\lambda)}{\lambda} + \psi(\lambda)\varphi_\theta(r)\varphi_\theta(\lambda) \qquad (4.8).$$

Inverting Laplace transforms in both sides of (4.8) the following defective renewal equation is got:

$$w_r(t;a,\theta) = \theta\varphi_a(r)F_a(t) + \int_0^t w_r(t-s;a,\theta)\varphi_\theta(r)f_\theta(s)ds \qquad (4.9).$$

Now an asymptotic approximation of $w_r(t;a,\theta)$ will be obtained through the key renewal theorem, see [16, pg. 376].

If in (4.9) $t \to \infty$

$$w_r(\infty;a,\theta) = \theta\varphi_a(r) + w_r(\infty;a,\theta)\varphi_\theta(r) \quad (4.10).$$

Or $w_r(\infty;a,\theta) = \frac{\theta\varphi_a(r)}{1-\varphi_\theta(r)} = v_r(a,\theta)$.

This is the expression (4.3) for $v_r(a,\theta)$. Subtracting each side of (4.10) from each side of (4.9), and performing some elementary calculations is obtained the following, still defective, renewal equation

$$J(t) = j(t) + \int_0^t J(t-s)\varphi_\theta(r)f_\theta(s)ds \qquad (4.11).$$

Here $J(t) = w_r(\infty;a,\theta) - w_r(t;a,\theta)$ and $j(t) = \theta\varphi_a(r)\big(1-F_a(t)\big) + \frac{\theta\varphi_a(r)\varphi_\theta(r)}{1-\varphi_\theta(r)}\big(1-F_\theta(t)\big)$.

Now, to obtain a common renewal equation from (4.11), it must be admitted the existence of a value $k > 0$ such that $\int_0^\infty e^{ks}\varphi_\theta(r)f_\theta(s)ds = \varphi_\theta(r)\varphi_\theta(-k) = 1$.

So, the transform $\varphi_\theta(\lambda)$ is defined in a domain different from the initially considered. That is, a domain including a convenient subset of the negative real numbers.

Multiplying both sides of (4.11) by e^{kt} the common renewal equation desired is finally obtained: $e^{kt}J(t) = e^{kt}j(t) + \int_0^t e^{k(t-s)}J(t-s)e^{ks}\,\varphi_\theta(r)f_\theta(s)ds$ from which, by the application of the key renewal theorem, it results

$$\lim_{t\to\infty} e^{kt}J(t) = \frac{1}{k_0}\int_0^\infty e^{ks}j(s)\,ds \quad (4.12).$$

And $k_0 = \int_0^\infty se^{ks}\,\varphi_\theta(r)f_\theta(s)ds = \varphi_\theta(r)\varphi_\theta'(-k)$, since $e^{kt}j(t)$ is directly Riemann integrable. The integral in (4.12) may be expressed in terms of transforms as $\int_0^\infty e^{ks}j(s)\,ds = \frac{\theta\varphi_a(r)\varphi_a(-k)}{k}$. So, an asymptotic approximation, in the sense of (4.12) was obtained:

$$w_r(t;a,\theta) \approx v_r(a,\theta) - c_r(a,\theta)e^{-k_r(\theta)t} \quad (4.13).$$

Here $k_r(\theta)$ is the positive value of k that fulfills:

$$\varphi_\theta(r)\varphi_\theta(-k) = 1 \qquad (4.14).$$

And

$$c_r(a,\theta) = \frac{\theta\varphi_a(r)\varphi_a(-k_r(\theta))}{-k_r(\theta)\varphi_\theta(r)\varphi_\theta'(-k_r(\theta))} \qquad (4.15).$$

5. Particularizing for Brownian Motion

Suppose the diffusion process $X(t)$, underlying the reserves value behavior of the pension's fund, is a generalized Brownian motion process, with drift $\mu(x) = \mu, \mu < 0$ and diffusion coefficient $\sigma^2(x) = \sigma^2, \sigma > 0$. Observe that the setting satisfies the conditions that were assumed above in this work. Namely $\mu < 0$ implies condition (4.1). Everything else remaining as previously stated, will be proceeded to present the consequences of this particularization. In general, it will be added $(*)$ to the notation used before because it is intended to use these specific results later.

To obtain the first passage time S_a Laplace transform, remember (4.2), it must be solved the equation: $\frac{1}{2}\sigma^2(a)u_\lambda^{*''}(a) + \mu(a)u_\lambda^{*'}(a) = \lambda u_\lambda^*(a), u_\lambda^*(a) = \varphi_a(\lambda), u_\lambda^*(0)$=1 $u_\lambda^*(\infty) = 0$. This is a homogeneous second order differential equation with constant coefficients, which general solution is $u_\lambda^*(a) = \beta_1 e^{\alpha_1 a} + \beta_2 e^{\alpha_2 a}$, with $\alpha_1, \alpha_2 = \frac{-\mu\pm\sqrt{\mu^2+2\lambda\sigma^2}}{\sigma^2}$.

Condition $u_\lambda^*(\infty) = 0$ implies $\beta_1 = 0$ and $u_\lambda^*(0)$=1 implies β_2=1 so that the solution is achieved:

$$u_\lambda^*(a) = e^{-K_\lambda a}\left(= \varphi_a^*(\lambda)\right), K_\lambda = \frac{\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2} \qquad (5.1).$$

In this case, the perpetual maintenance cost present value of the pensions fund is given by, following (4.1) and using (5.1),

$$v_r^*(a,\theta) = \frac{\theta e^{-K_r a}}{1 - e^{-K_r\theta}} \qquad (5.2).$$

Note that $v_r^*(a,\theta)$ is a decreasing function of the first variable and an increasing function of the second. Proceeding as before:

$$\lim_{\theta\to 0} v_r^*(a,\theta) = \frac{e^{-K_r a}}{K_r} \qquad (5.3).$$

This expression has been obtained in [6] , in a different context and using different methods but, obviously, with identical significance. In [6], the authors acted with a generalized Brownian motion, with no constraints in what concerns the drift coefficient, conditioned by a reflection scheme at the origin.

A way to reach an expression for the finite period maintenance cost present value, is starting by the computation of $k_r^*(\theta)$, solving (4.14). This means to determine a positive number k satisfying $e^{-K_r\theta}e^{-K_{-\lambda}\theta} = 1$ or $K_r + K_{-\lambda} = 0$.

This identity is verified for the value of k:

$$k_r^*(\theta) = \frac{\mu^2 - \left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)^2}{2\sigma^2}, \text{if } \mu < -\sqrt{\frac{2r\sigma^2}{3}} \quad (5.4).$$

Note that the solution is independent of θ in these circumstances. A simplified solution, independent from a and θ, for $c_r^*(a,\theta)$ was also obtained. Using (4.15) the result is

$$c_r^*(a,\theta) = \frac{2\sigma^2\left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)}{\mu^2 - \left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)^2} \quad (5.5).$$

Combining these results, (5.4) and (5.5), as in (4.13) it is observable that the asymptotic approximation for this particularization reduces to $w_r^*(t;a,\theta) \approx v_r^*(a,\theta) - \pi_r(t)$, where the function $\pi_r(t)$ is, considering (5.4) and (5.5),

$$\pi_r(t) = \frac{2\sigma^2\left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)}{\mu^2 - \left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)^2} e^{-\frac{\mu^2 - \left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)^2}{2\sigma^2}t}, \text{if } \mu < -\sqrt{\frac{2r\sigma^2}{3}} \quad (5.6).$$

6.The Assets and Liability Behavior Representation

In this section it is presented an application of the results obtained above to an asset-liability management scheme of a pension fund. Assume that the assets value process of a pensions fund may be represented by the geometric Brownian motion process

$$A(t) = be^{a+(\rho+\mu)t+\sigma B(t)} \text{ with } \mu < 0 \text{ and } ab\rho + \mu\sigma > 0 \quad (6.1),$$

where $B(t)$ is a standard Brownian motion process. Suppose also that the fund liabilities value process performs such as the deterministic process $L(t) = be^{\rho t}$.

Consider now the stochastic process $Y(t)$ obtained by the elementary transformation of $A(t)$, $Y(t) = \ln\frac{A(t)}{L(t)} = a + \mu t + \sigma B(t)$. This is a generalized Brownian motion process exactly as the one studied before, starting at a, with drift μ and diffusion coefficient σ^2. Note also that the firs passage time of the assets process $A(t)$ by the mobile barrier T_n, the liabilities process, is the first passage time of $Y(t)$ by 0-with finite expected time under the condition, stated before, $\mu < 0$.

Consider also the pensions fund management scheme that raises the assets value by some positive constant θ_n, when the assets value falls equal to the liabilities process by the n^{th} time. This corresponds to consider the modification $\overline{A}(t)$ of the process $A(t)$ that restarts at times T_n when $A(t)$ hits the barrier $L(t)$ by the n^{th} time at the level $L(T_n) + \theta_n$. For purposes of later computations, it is a convenient choice the management policy where

$$\theta_n = L(T_n)\left(e^{\theta} - 1\right), \text{for some } \theta > 0 \quad (6.2).$$

The corresponding modification $\widetilde{Y}(t)$ of $Y(t)$ will behave as a generalized Brownian motion process that restarts at the level $\ln\frac{L(T_n)+\theta_n}{L(T_n)} = \theta$ when it hits 0 (at times T_n).

Proceeding this way, it is reproduced via $\widetilde{Y}(t)$ the situation observed before when the Brownian motion example was treated. The Laplace transform (5.1) is still valid.

Pensions Fund Perpetual Maintenance Cost Present Value Expectation

Similarly, to former proceedings, the results for the present case will be distinguished with the symbol $(\#)$. It is considered the pensions fund perpetual maintenance cost present value, because of the proposed asset-liability management scheme, given by the random variable:$V^{\#}(r,a,\theta)=\sum_{n=1}^{\infty}\theta_n e^{-rT_n}=\sum_{n=1}^{\infty}b\left(e^{\theta}-1\right)e^{-(r-\rho)T_n}, r>\rho$, where r represents the appropriate discount interest rate. To obtain the above expression it was only made use of the $L(t)$ definition and (6.1). Note that it is possible to express the expected value of the above random variable with the help of (6.2) as

$$v_r^{\#}(a,\theta)=\frac{b\left(e^{\theta}-1\right)}{\theta}v_{r-\rho}^{*}(a,\theta)=\frac{b\left(e^{\theta}-1\right)e^{-K_{r-\rho}a}}{1-e^{-K_{r-\rho}\theta}} \qquad (6.3).$$

As $\theta\to 0$

$$\lim_{\theta\to 0}v_r^{\#}(a,\theta)=\frac{be^{-K_{r-\rho}a}}{K_{r-\rho}} \quad (6.4).$$

Another expression that may be found in $[6]$.

Maintenance Cost up to Time t Expected Value

In a similar way, the maintenance cost up to time t in the above-mentioned management scheme, is the stochastic process$W^{\#}(t;r,a,\theta)=\sum_{n=1}^{N(t)}b\left(e^{\theta}-1\right)e^{-(r-\rho)T_n}$, $W^{\#}(t;r,a,\theta)=0$ if $N(t)=0$, with expected value function

$$w_r^{\#}(t;a,\theta)=\frac{b\left(e^{\theta}-1\right)}{\theta}w_{r-\rho}^{*}(t;a,\theta) \qquad (6.5).$$

7.Conclusions

We initially started by establishing an equilibrium condition for pension funds, using a queuing network model with infinite servers at each node. In this condition, financial and demographic aspects were considered.

Then, with a view to rebalancing unbalanced funds, through external financing, we studied its modeling using diffusion processes

In the general diffusion scenery, the main results are formulae (4.3) and (4.13). The whole work depends on the possibility of solving equation (4.2) to obtain the first passage times Laplace transforms. Unfortunately, the solutions are known only for rare cases. An obvious case for which the solution of the equation is available is the one of the Brownian motion diffusion process. The main results concerning this particularization are formulae (5.2) and (5.6). Some transformations of the Brownian motion process that allowed to make use of the available Laplace transform may be explored as it was done in section 6. Formulae (6.3) and (6.5) are this application most relevant results.

In $[20, \text{and } 21]$ an alternative approach based on gambler's ruin problem is presented.

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Competing Interests

Authors have declared that no competing interests exist.

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