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A Moving Optimal Control Problem for a Parabolic System

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***Abstract*— The paper deals with a moving optimal control problem for heat conductivity processes. A quadratic functional is taken as an optimality criterion. The existence of the stated problem is studied. Definition of the the optimal control is given.**

Keywords— moving optimal control, functional, a class of admisible controls, Dirac function

I. Introduction

The paper deals with a moving optimal control problem for heat-conductivity processes. The problem is: it is required to find such a control $p(t)=\{p_1(t),p_2(t),\ldots,p_{m-1}(t)\}\in U$ from the class of possible controls that affords a minimum value to the functional

$$I(p)=\int_0^{\ell} u^2(x,t)dx$$

within the solution of the problem (1)-(3) satisfying the initial boundary conditions.

The solution of the stated mixed problem (1)-(3) for each fixed control at first is sought in the form of the solution

$$u(x,t)=X(x)T(t)$$

satisfying boundary conditions (1) and initial conditons (3) of the homogeeneous equation corresponding to the equation (1), so, the functions $X(x)T(t)$ are non-trivial functions [1].

At first we find the solution of equation (1) satisfying the initial and homogeneous conditions [2]. Then according to the known rule, the solution of the stated mixed problem for each fixed control is in the form $u(x,t)$ [2].

II. Problem Statement

Let the temperature at the ends of the rod of length ℓ be equal zero and the rod is supplied with heat of intensity $p_1(t),p_2(t),\ldots,p_{m-1}(t)$ at the points $0<x_1<\ldots<x_{m-1}<\ell$ outside. Then this process is mathematically described bu the equation

$$\rho(x)\frac{\partial u(x,t)}{\partial t}=\frac{\partial}{\partial x}[a(x)\frac{\partial u(x,t)}{\partial x}]+\sum_{i=1}^{m-1}P_i(t)\delta(x-x_i), \quad (1)$$

boundary conditions

$$u(0,t)=0,\ u(\ell,0)=0,\ 0<t\le T, \quad (2)$$

and initial conditons

$$u(x,0)=\varphi(x),\ \ 0\le x<\ell. \quad (3)$$

Here $\rho(x)$ is the density of the rod material and is positive in the interval $0\le x\le \ell$, the function $a(x)$ is a given function differentiable in the interval $(0,\ell)$, $\varphi(x)$ is a continuous function in the interval $[0,\ell]$, $p_1(t),p_2(t),\ldots,p_{m-1}(t)$ are control functions and are taken from the class of possible controls

$$U=\{p_i(t):\int_0^{\ell} p_i^2(t)dt\le L_i;\ i=\overline{1,m-1}\}$$

δ - is Driac's "delta" function.

The problem is: to find such a control $p(t)=\{p_1(t),p_2(t),\ldots,p_{m-1}(t)\}\in U$ from the class of posible functions that affords a minimum to the functional

$$J(p)=\int_0^{\ell} u^2(x,T)dx \quad (4)$$

within the solution of the problem (1)-(3).

2.The solution of the mixed problem (1)-(3) for each fixed control at first is found in the form of homogeneous equation

$$\rho(x)\frac{\partial u(x,t)}{\partial t}=\frac{\partial}{\partial x}\left[a(x)\frac{\partial u(x,t)}{\partial x}\right] \quad (5)$$

corresponding to the equation (1), the solution satisfying boundary conditions (2) and initial conditions (3) in the form

$$u(x,t)=X(x)T(t) \quad (6)$$

so, the functions $X(x)T(t)$ are non-trivial functions.

Since

$$\frac{\partial u(x,t)}{\partial t} = X(x)T'(t),\ \frac{\partial}{\partial x}[a(x)\frac{\partial u(x,t)}{\partial x}] = \frac{d}{dx}[a(x)\frac{dX(x)}{dx}]T(t)$$

from the equation (5)

$$\rho(x)X(x)T(t) \equiv \frac{d}{dx}[a(x)\frac{dX(x)}{dx}]T(t),$$

Hence we obtain

$$\frac{T'(t)}{T(t)} \equiv \frac{\frac{d}{dx}[a(x)\frac{dX(x)}{dx}]}{\rho(x)X(x)} = -\lambda$$

Thus, for the function determined by the equality (6) be the solution of the equation (5) it is necessary functions $T(t)$ and $X(x)$ recpectivelly be the solutions of the following equation:

$$T'(t) + \lambda T(t) = 0, \tag{7}$$

$$\frac{d}{dx}\left[a(x)\frac{dX(x)}{dx}\right] + \lambda\rho(x)X(x) = 0, \tag{8}$$

For boundary conditions (2) be satisfied, it is necessary

$$X(0) = 0,\ X(\ell) = 0 \tag{9}$$

When $a(x)$ and $\rho(x)$ satisfy the above conditions, the spectral problem (8), (9) has an increasing sequence of eigen-values $\{\lambda_k\}$ with the limit $+\infty$ and a system of eigen-functions $\{X_k(x)\}$ orthonormal in the interval $[0,\ell]$.

Writing $\lambda = \lambda_n$ in the equation (7), the solution of the obtained equation is the form $T_n(t) = c_n e^{-\lambda_n t}$. So, the solution of the problem (5), (2)is in the form

$$u^*(x,t) = \sum_{n=1}^{\infty} \varphi_n e^{-\lambda_n t} X_n(x), \tag{10}$$

Since

$$\sum_{i=1}^{m-1} p_i(t)\delta(x - x_i) \in L_2[\delta \le x \le \ell,\ 0 \le t \le T]$$

we can expand it in Fourier series in $[0,\ell]$ with respect to the orthonormal system $\{X_n(x)\}$ i.e. the expansion

$$\sum_{i=1}^{m-1} p_i(t)\delta(x - x_i) = \sum_{i=1}^{m-1} b_n(t)X_n(x),$$

$$b_n(t) = \int_0^{\ell} \sum_{i=1}^{m-1} p_i(t)\delta(x - x_i)X_n(x)dx =$$

$$\sum_{i=1}^{m-1} p_i(t)\int_0^{\ell} \delta(x - x_i)X_n(x)dx = \sum_{i=1}^{m-1} p_i(t)X_n(x_i)$$

is valid.

In can be easily shown that the solution of the equation (1) satisfying the homogeneous initial condition and homogeneous boundary conditions is in the form

$$\bar{u}(x,t) = \sum_{n=1}^{\infty}\int_0^{\ell}\sum_{i=1}^{m-1} p_i(\tau)X_n(x_i)\ell^{-\lambda_n\tau}d\tau X_n(x)$$

Then acording to the known rule, the solution of the problem (1)-(3) for each fixed control is in the form

$$u(x,t) = \sum_{n=1}^{\infty}\left[\varphi_n + \int_0^{\ell}\sum_{i=1}^{m-1} p_i(\tau)X_n(x_i)\ell^{\lambda_n\tau}d\tau\right]\ell^{-\lambda_n t}X_n(x), \tag{11}$$

It shoul be noted that the function determined by the equality (11) is the generalized solution of the problem (1)-(3).

III. Solutİon of the optİmal control problem

Having substituted the solution of the problem (1)-(3) determined by the equality (11) in the expression of the functional $J(p)$ and taking into account that the system $\{X_n(x)\}$ is orthonormal in the interval $[0,\ell]$ and making the following replacement, we obtain:

$$I = \sum_{n=1}^{\infty}\varphi_n^2\ell^{-\lambda_n T},$$

$$\omega_i(\tau) = \sum_{n=1}^{\infty}\varphi_n X_n(x_i)\ell^{-\lambda_n(2T-\tau)}$$

$$R_{ij}(\tau,s) = \sum_{n=1}^{\infty} X_n(x_i)X_n(x_j)\ell^{-\lambda_n(2T-\tau-s)}$$

After this replacement, we can write the functional $J(p)$ as follows

$$J(p) = I + 2\int_0^T\sum_{i=1}^{m-1}\omega_i(\tau)p_i(\tau)d\tau + \\ + \int_0^T\int_0^T\sum_{i,j=1}^{m-1} R_{ij}(t,s)p_i(t)p_j(s)dtds. \tag{12}$$

Theorem. The stated problem has a solution if the conditions

$a(x) \in C^1(0,\ell),\ \varphi(x) \in C(0,\ell),\ \rho(x) \in C(0,\ell),\ \rho(x) > 0,$

$x \in (0,\ell)$, $p_i(t) \in L_2(0,T),\ i = 1,\dots,m-1$ are satisfied.

IV. Conclusion

In the paper we study an optimal control problem for a system described by a parabolic type equation

$$\rho(x)\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[a(x)\frac{\partial u(x,t)}{\partial x}\right] + \sum_{i=1}^{m-1} p_i(t)\delta(x - x_i)$$

Quadratic functional is taken as an optimality criterion [2]. At first we define the solution to the mixed problem for each control. Then a theorem on the existence and uniquness of the optimal control is proved [3].

References

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