A Formula that Generates the Sum of First n Factorial

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ABSTRACT:

The author proposes to find a generic formula for the sum of first n factorials (i.e. $\sum_{k=1}^{n} k!$). Also, the author puts to use Ramanujan approximation of factorial to create a variant for the factorial sum.

I. Introduction

Divergent series have been the nightmare of mathematicians from a very long time, yet they are also a place of perennial amusement.
As Abel remarked in a letter to Holmboe, Jan 1826[1]:

"Divergent series are in general something fatal, and it is a disgrace to base any proof in them." Often translated as "Divergent series are an invention of the devil…"

We consider the factorial series here. The following sequence, mentioned in the Online Encyclopedia of Integer Sequences (OEIS A000142) [2] is wildly divergent:

$$1!, 2!, 3!, 4!, \dots \dots \dots . n!, (n+1)! \dots \dots ..$$

In this paper, we propose to find a formula that gives the sum of first n factorials. That is,

$$\sum_{k=0}^{n} k! = 0! + 1! + 2! + 3! + \cdots \dots \dots + n! \qquad \S 1$$

This sum doesn't actually seem to give away easily, but there exists a clever trick that does the job for us.

II. The Idea

The following series is very well known as the geometric series:

$$\sum_{k=0}^{n} b^k = 1 + b + b^2 + b^3 + \cdots .. + b^n \qquad \text{§2.1}$$

We have also known that this expression is equal to:

$$\sum_{k=0}^{n} b^k = \frac{a(1-b^{n+1})}{1-b} \qquad \text{§2.2}$$

Where a is the first term in series. Here,$a = 1$. We use this fact in a neat way to give us our desired results. The idea is, we take the Laplace Transform on the both sides, which works like magic. This will be demonstrated in the next section. But before that we should note that it satisfies all the conditions for the existence of the transform, namely:

1. The function $f(x)$is continuous or piecewise continuous in the closed interval $[a, b]$ where $a > 0$.
2. It is of exponential order; and
3. $x^n f(x)$is bounded near $x = 0$ for some number $n > 1$.

III. The Formula

Now consider §2.2 . The sum is: $\sum_{k=0}^{n} b^k = \frac{(1-b^{n+1})}{1-b}$

Now, take the Laplace Transform on the both sides:

$$\mathcal{L}(\sum_{k=0}^{n} b^k) = \mathcal{L}\left(\frac{a(1-b^{n+1})}{1-b}\right)$$

Or, $\sum_{k=0}^{n} \mathcal{L}(b^k) = \mathcal{L}\left(\frac{a(1-b^{n+1})}{1-b}\right)$

It is a well-known fact that $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

Thus, on substitution the sum becomes:

$$\sum_{k=0}^{n} \frac{k!}{s^{k+1}} = \mathcal{L}\left(\frac{a(1-b^{n+1})}{1-b}\right) \qquad §3.1$$

Now, by the virtue of definition, the right hand side takes the form:

$$\sum_{k=0}^{n} \frac{k!}{s^{k+1}} = \int_0^{\infty} \frac{1-b^{n+1}}{1-b} e^{-sb} db \qquad §3.2$$

Finally let $s = 1$ to obtain:

$$\sum_{k=0}^{n} k! = \int_0^{\infty} \frac{1-b^{n+1}}{1-b} e^{-b} db \qquad §3.3$$

This is the required formula for the sum of first n factorials.

IV. Corollary

An interesting formula can be deduced from this formula and the approximation for $n!$ given by Srinivasa Ramanujan[3].
We know that

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1+\frac{1}{2n}+\frac{1}{8n^2}+\cdots..\right)^{1/6}$$

Thus if we were to substitute this in §3.3, we will obtain:

$$\sum_{k=0}^{n} k! \approx \sum_{k=0}^{n} \sqrt{2\pi k}\left(\frac{k}{e}\right)^k \left(1+\frac{1}{2k}+\frac{1}{8k^2}+\cdots...\right)^{1/6}$$

Simplifying which yields us:

$$\sum_{k=0}^{n} k! \approx \sqrt{2\pi}\sum_{k=0}^{n} \frac{k^{2k+1/2}}{e^k}\left(1+\frac{1}{2k}+\frac{1}{8k^2}+\cdots.\right)^{1/6} \qquad §3.4$$

V. Examples & Notes

We will illustrate some cases to show the use of formula §3.3.

1. For $n = 0$

   We have, $\sum_{k=0}^{0} k! = \int_0^\infty \frac{1-b}{1-b} e^{-b} db = \int_0^\infty e^{-b} db$

   $= \mathrm{e}^0 + \lim_{\mathrm{n}\to\infty} (-\mathrm{e}^{-\mathrm{b}}) = 1$

2. For n=1

   We have, $\sum_{k=0}^{0} k! = \int_0^\infty \frac{1-b^2}{1-b} e^{-b} db = \int_0^\infty (1+b) e^{-b} db$

   $= 2$

And so on. In case of large numbers (like 60 or so) it may be found that numerical integration is at our disposal, which does the task smoothly, without any hitch.

VI. Conclusion

We have established the formulae

$$\sum_{k=0}^{n} k! = \int_{0}^{\infty} \frac{1-b^{n+1}}{1-b} e^{-b} db$$

$$\sum_{k=0}^{n} k! \approx \sqrt{2\pi} \sum_{k=0}^{n} \frac{k^{2k+1/2}}{e^k} \left(1 + \frac{1}{2k} + \frac{1}{8k^2} + \cdots ..\right)^{1/6}$$

for the sum of first n factorials.

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VIII. References

[1]. https://en.wikipedia.org/wiki/Divergent_series
[2].https://oeis.org/A000142
[3]. https://en.wikipedia.org/wiki/Stirling%27s_approximation