Near-Square Primes Conjecture

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ABSTRACT. In 1912, Edmund Landau listed four basic problems about prime numbers in the International Congress of Mathematicians. These problems are now known as Landau's problems. Landau's fourth problem asked whether there are infinitely many primes which are of the form $n^2 + 1$ for some integer n. This problem remains open and it is known as the Near-square primes conjecture. We prove this conjecture is indeed true.

1. RESULTS

Definition 1.1. Given a function $f : \mathbb{N} \to \mathbb{R}$, we define

$$\lim_{n\to\infty} I(f(n)) = 1$$

when $\exists m_0 \in \mathbb{N}$ such that $\forall n > m_0 : f(n) \in \mathbb{Z}$ and

$$\lim_{n\to\infty} I(f(n)) = 0$$

when $\nexists m_0 \in \mathbb{N}$ such that $\forall n > m_0 : f(n) \in \mathbb{Z}$.

Lemma 1.2. *Given a function $f : \mathbb{N} \to \mathbb{R}$ and an irrational number α, we have*

$$\lim_{n\to\infty} I(\alpha \times f(n)) = 0$$

when

$$\lim_{n\to\infty} I(f(n)) = 1.$$

Proof. Certainly, a number $\alpha \times k$ is not an integer when k is an integer even though k could be no matter how large we want. $\square$

Theorem 1.3. *There are infinitely many primes which are of the form $n^2 + 1$ for some integer n.*

Proof. Suppose, there are not infinitely many primes which are of the form n^2+1 for some integer n. In number theory, Wilson's theorem states that a natural number $n > 4$ is a composite number if and only if the product of all the positive integers less than n is multiple of n [1]. That is the factorial $(n-1)! = 1\times2\times3\times\cdots\times(n-1)$ satisfies

$$(n-1)! \equiv 0 \pmod n$$

exactly when n is a composite number [1]. In this way, if the Near-square primes conjecture is false, then we would have that $n^2 + 1$ must be a composite number when n tends to infinity. Consequently, we obtain

$$\lim_{n\to\infty} I(\frac{n^2!}{n^2+1}) = 1.$$

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We know

$$\prod_{j=1}^{\infty} \frac{(p_j^2 - 1)}{(p_j^2 - 1)} = 1$$

where p_j is the j^{th} prime number. We also know

$$\lim_{n\to\infty} I(\frac{n^2!}{n^2+1} \times 1) = 1$$

and thus, we obtain

$$\lim_{n\to\infty} I(\frac{n^2!}{n^2+1} \times \prod_{j=1}^{\infty} \frac{(p_j^2 - 1)}{(p_j^2 - 1)}) = 1.$$

In this way, we have

$$\lim_{n\to\infty} I(\frac{n^2!}{n^2+1} \times \prod_{j=1}^{\infty} \frac{(p_j^2 - 1)}{(p_j^2 - 1)}) = 1$$

is equivalent to

$$\lim_{n\to\infty} I(\prod_{j=1}^{\infty}(p_j^2) \times g(n) \times \prod_{j=1}^{\infty} \frac{(p_j^2 - 1)}{(p_j^2 - 1)}) = 1$$

where

$$\lim_{n\to\infty} I(g(n)) = 1$$

since there is no square of a prime number p_j^2 that must necessarily be eliminated from the division $\frac{n^2!}{n^2+1}$ in the numerator $n^2!$. In addition, we can transform this limit into

$$\lim_{n\to\infty} I(\prod_{j=1}^{\infty} \frac{p_j^2}{p_j^2 - 1} \times h(n)) = 1$$

where

$$\lim_{n\to\infty} I(h(n)) = \lim_{n\to\infty} I(g(n) \times \prod_{j=1}^{\infty}(p_j^2 - 1)) = \lim_{n\to\infty} I(g(n) \times \prod_{p_j < n^2+1}(p_j^2 - 1)) = 1$$

since all the prime numbers p_j are lesser than $n^2 + 1$ when n tends to infinity. However,

$$\lim_{n\to\infty} I(\prod_{j=1}^{\infty} \frac{p_j^2}{p_j^2 - 1} \times h(n)) = 1$$

would be the same as

$$\lim_{n\to\infty} I(\frac{\pi^2}{6} \times h(n)) = 1$$

since we have

$$\prod_{j=1}^{\infty} \frac{p_j^2}{p_j^2 - 1} = \prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-2}} = \frac{\pi^2}{6}$$

as a consequence of the result in the Basel problem [1]. Hence, we obtain a contradiction since

$$\lim_{n\to\infty} I(\frac{\pi^2}{6} \times h(n)) = 0$$

according to the Lemma 1.2. To sum up, we have that our assumption that the Near-square primes conjecture were false is incorrect and therefore, we obtain the conjecture should be necessarily true. □

References

[1] David G. Wells. *Prime Numbers, The Most Mysterious Figures in Math*. John Wiley & Sons, Inc., 2005.

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