Proposed Proof of the Riemann Hypothesis

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Abstract. For every prime number q_n, we define the inequality $\prod_{q \leq q_n} \frac{q}{q-1} > e^{\gamma} \times \log \theta(q_n)$, where $\theta(x)$ is the Chebyshev function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. This is known as the Nicolas inequality. The Nicolas criterion states that the Riemann hypothesis is true if and only if the Nicolas inequality is satisfied for all primes $q_n > 2$. We prove indeed that the Nicolas inequality is satisfied for all primes $q_n > 2$. In this way, we show that the Riemann hypothesis is true.

1. INTRODUCTION The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [**1**]. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

where $q \leq x$ means all the prime numbers q that are less than or equal to x. We define a sequence based on this function.

Definition. For every prime number q_n, we define the sequence of real numbers:

$$X_n = \frac{\prod_{q \leq q_n} \frac{q+1}{q}}{\log \theta(q_n)}.$$

Say $\mathsf{Nicolas}(q_n)$ holds provided

$$\prod_{q \leq q_n} \frac{q}{q-1} > e^{\gamma} \times \log \theta(q_n).$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and $\log$ is the natural logarithm. The importance of this inequality is:

Theorem 1. *[**4**].* $\mathsf{Nicolas}(q_n)$ *holds for all prime numbers* $q_n > 2$ *if and only if the Riemann hypothesis is true.*

We use this limit superior,

Theorem 2. *[**2**].*

$$\limsup_{n \to \infty} X_n = \frac{e^{\gamma} \times 6}{\pi^2}.$$

Besides, we define the following value,

Theorem 3. *[**3**].*

$$\prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \zeta(2) = \frac{\pi^2}{6}.$$

Putting all together yields the proof that the Nicolas inequality is satisfied for all prime numbers greater than 2. Consequently, we prove that the Riemann hypothesis is true.

2. A CENTRAL LEMMA The following is a key lemma.

Lemma 1. *There exists a natural number N such that $X_n < \frac{e^\gamma \times 6}{\pi^2} + \varepsilon$ for all natural numbers $n > N$ and $\varepsilon \leq \frac{6}{\pi^2}$.*

Proof. The limit superior of a sequence of real numbers y_n is the smallest real number b such that, for any positive real number ε, there exists a natural number N such that $y_n < b + \varepsilon$ for all natural numbers $n > N$. Therefore, this is a consequence of the theorem 2. ■

3. PROOF OF MAIN THEOREM

Theorem 4. *The Riemann hypothesis is true.*

Proof. From the lemma 1, we know that there exists a natural number N such that $X_n < \frac{e^\gamma \times 6}{\pi^2} + \varepsilon$ for all natural numbers $n > N$ and $\varepsilon \leq \frac{6}{\pi^2}$. We multiply the both sides of the inequality

$$\frac{\prod_{q \leq q_n} \frac{q+1}{q}}{\log \theta(q_n)} < \frac{e^\gamma \times 6}{\pi^2} + \varepsilon$$

by

$$\prod_{q \leq q_n} \frac{q^2}{q^2 - 1}$$

to obtain that

$$\frac{\prod_{q \leq q_n} \frac{q}{q-1}}{\log \theta(q_n)} < \prod_{q \leq q_n} \frac{q^2}{q^2 - 1} \times \frac{6}{\pi^2} \times (e^\gamma + c)$$

for the constant $c = \varepsilon \times \frac{\pi^2}{6}$ due to

$$\frac{q}{q-1} = \frac{q^2}{q^2 - 1} \times \frac{q+1}{q}.$$

From the theorem 3, we note that $\prod_{q \leq q_n} \frac{q^2}{q^2-1} \times \frac{6}{\pi^2}$ is strictly increasing as q_n increases. Besides, we have that

$$\lim_{n \to \infty} \frac{6}{\pi^2} \times \left(\prod_{q \leq q_n} \frac{q^2}{q^2 - 1} \right) = 1.$$

Proposition. We state the following proposition S: There exists a natural number m such that

$$\prod_{q \leq q_m} \frac{q^2}{q^2 - 1} \times \frac{6}{\pi^2} \leq \frac{e^\gamma}{e^\gamma + c}$$

for a sufficiently small constant $0 < c \leq 1$.

Hence, we could have

$$\frac{\prod_{q \leq q_m} \frac{q}{q-1}}{\log \theta(q_m)} < \frac{e^{\gamma} \times d}{e^{\gamma} + c} \times (e^{\gamma} + c)$$

for some small constant $0 < d \leq 1$. This implies that

$$\prod_{q \leq q_m} \frac{q}{q-1} < (e^{\gamma} \times d) \times \log \theta(q_m) \leq e^{\gamma} \times \log \theta(q_m).$$

Hence, $\mathsf{Nicolas}(q_m)$ would not hold.

Proposition. We state another proposition T: *The Riemann hypothesis is false*.

So, we would have the implication $S \Rightarrow T$ should be true because of the theorem 1. However, we know that

$$\prod_{q \leq 3} \frac{q^2}{q^2-1} \times \frac{6}{\pi^2} \leq \prod_{q \leq q_m} \frac{q^2}{q^2-1} \times \frac{6}{\pi^2}$$

and thus, we would get

$$\prod_{q \leq 3} \frac{q^2}{q^2-1} \times \frac{6}{\pi^2} \leq \frac{e^{\gamma}}{e^{\gamma} + c}.$$

Following the previous steps, we would obtain that $\mathsf{Nicolas}(3)$ does not hold. In this way, we obtain a contradiction since $\mathsf{Nicolas}(3)$ holds indeed. Consequently, the implication $S \Rightarrow T$ cannot be true. If the implication $S \Rightarrow T$ is false, then T is also false. So, the proposition T which exactly states that:

The Riemann hypothesis is false

cannot be true. By contraposition, we can conclude that the Riemann hypothesis is indeed true. ■

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