Short Note about the Robin's Inequality

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Abstract. In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics. The Robin's inequality consists in $\sigma(n) < e^{\gamma} \times n \times \ln\ln n$ where $\sigma(n)$ is the divisor function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The Robin's inequality is true for every natural number $n > 5040$ if and only if the Riemann hypothesis is true. Given a natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m}$ such that $n > 5040$, $q_1, q_2, \cdots, q_m$ are prime numbers and $a_1, a_2, \cdots, a_m$ are positive integers, then the Robin's inequality is true for n when $q_1^{\alpha} \times q_2^{\alpha} \times \cdots \times q_m^{\alpha} \leq n$, where $\alpha = (\ln n')^{\beta}$, $\beta = (\frac{\pi^2}{6} - 1)$ and n' is the squarefree kernel of n.

1. Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics [2]. It is of great interest in number theory because it implies results about the distribution of prime numbers [2]. It was proposed by Bernhard Riemann (1859), after whom it is named [2]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution [2].

The divisor function $\sigma(n)$ for a natural number n is defined as the sum of the powers of the divisors of n

$$\sigma(n) = \sum_{k \mid n} k$$

where $k \mid n$ means that the natural number k divides n [3]. In 1915, Ramanujan proved that under the assumption of the Riemann hypothesis, the inequality

$$\sigma(n) < e^{\gamma} \times n \times \ln\ln n$$

holds for all sufficiently large n, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant [2]. The largest known value that violates the inequality is $n = 5040$. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann hypothesis is true [2]. Using this inequality, we show an interesting result.

2. Results

Theorem 2.1. *Given a natural number*

$$n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m}$$

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such that $q_1, q_2, \cdots, q_m$ are prime numbers and $a_1, a_2, \cdots, a_m$ are positive integers, then we obtain the following inequality

$$\frac{\sigma(n)}{n} < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

Proof. From the article reference [1], we know that

$$\frac{\sigma(n)}{n} < \prod_{i=1}^{m} \frac{q_i}{q_i - 1}. \tag{2.1}$$

We can easily prove that

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} = \prod_{i=1}^{m} \frac{1}{1 - q_i^{-2}} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

However, we know that

$$\prod_{i=1}^{m} \frac{1}{1 - q_i^{-2}} < \prod_{j=1}^{\infty} \frac{1}{1 - q_j^{-2}}$$

where q_j is the j^{th} prime number and

$$\prod_{j=1}^{\infty} \frac{1}{1 - q_j^{-2}} = \frac{\pi^2}{6}$$

as a consequence of the result in the Basel problem [3]. Consequently, we obtain that

$$\frac{\sigma(n)}{n} < \prod_{i=1}^{m} \frac{q_i}{q_i - 1} < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

□

Theorem 2.2. *Given a natural number*

$$n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} > 5040$$

such that $a_1, a_2, a_3 \geq 0$ are integers, then the Robin's inequality is true for n.

Proof. Given a natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} > 5040$ such that $a_1, a_2, a_3 \geq 0$ are integers, we need to prove that

$$\frac{\sigma(n)}{n} < e^{\gamma} \times \ln \ln n$$

that is true when

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times \ln \ln n$$

according to the inequality (2.1). Given a natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} > 5040$ such that $a_1, a_2, a_3 \geq 0$ are integers, we have that

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^{\gamma} \times \ln \ln(5040) \approx 3.81.$$

However, we know for $n > 5040$ that

$$e^{\gamma} \times \ln \ln(5040) < e^{\gamma} \times \ln \ln n$$

and therefore, the proof is completed. □

Definition 2.3. We recall that an integer n is said to be squarefree if for every prime divisor q of n we have $q^2 \nmid n$, where $q^2 \nmid n$ means that q^2 does not divide n [1].

Theorem 2.4. *Given a natural number*

$$n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m}$$

such that $n > 5040$, $q_1, q_2, \cdots, q_m$ *are prime numbers and* $a_1, a_2, \cdots, a_m$ *are positive integers, then the Robin's inequality is true for* n *when* $q_1^{\alpha} \times q_2^{\alpha} \times \cdots \times q_m^{\alpha} \leq n$, *where* $\alpha = (\ln n')^{\beta}$, $\beta = (\frac{\pi^2}{6} - 1)$ *and* n' *is the squarefree kernel of* n.

Proof. We will check the Robin's inequality for every natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that $q_1, q_2, \cdots, q_m$ are prime numbers and $a_1, a_2, \cdots, a_m$ are positive integers. We need to prove that

$$\frac{\sigma(n)}{n} < e^{\gamma} \times \ln \ln n$$

that is true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \leq e^{\gamma} \times \ln \ln n$$

according to the Theorem 2.1. From a squarefree number n', we obtain that

$$\sigma(n') = (q_1 + 1) \times (q_2 + 1) \times \cdots \times (q_m + 1) \tag{2.2}$$

when $n' = q_1 \times q_2 \times \cdots \times q_m$ [1]. Using the equation (2.2), we obtain that will be equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^{\gamma} \times \ln \ln n$$

where $n' = q_1 \times \cdots \times q_m$ is the squarefree kernel of n [1]. However, the Robin's inequality has been proved for all the squarefree integers $n' \notin \{2, 3, 5, 6, 10, 30\}$ [1]. In addition, according to the Theorem 2.2, the Robin's inequality is true for every natural number $n > 5040$ when $n' \in \{2, 3, 5, 6, 10, 30\}$, where n' is the squarefree kernel of n. In this way, we have that

$$\frac{\sigma(n')}{n'} < e^{\gamma} \times \ln \ln n'$$

and therefore, it is enough to prove that

$$\frac{\pi^2}{6} \times e^{\gamma} \times \ln \ln n' \leq e^{\gamma} \times \ln \ln n$$

which is the same as

$$\frac{\pi^2}{6} \times \ln \ln n' \leq \ln \ln n$$

and

$$\ln(\ln n')^{\frac{\pi^2}{6}} \leq \ln \ln n$$

that is true when

$$(\ln n')^{\frac{\pi^2}{6}} \leq \ln n$$

is true. Consequently, that would be equivalent to

$$e^{(\ln n')^{\frac{\pi^2}{6}}} \leq n$$

which is equal to

$$e^{(\ln n') \times (\ln n')^{(\frac{\pi^2}{6} - 1)}} \leq n$$

that is

$$n'^{(\ln n')^{(\frac{\pi^2}{6}-1)}} \leq n$$

and thus, the proof is completed. □

3. Conclusions

The practical uses of the Riemann hypothesis include many propositions which are known true under the Riemann hypothesis, and some that can be shown equivalent to the Riemann hypothesis [2]. Certainly, the Riemann hypothesis is close related to various mathematical topics such as the distribution of prime numbers, the growth of arithmetic functions, the Lindelöf hypothesis, the large prime gap conjecture, etc [2]. Indeed, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as the number theory and pure mathematics [2]. In this way, we made a new step forward in the efforts of trying to prove the Riemann hypothesis.

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