On the Tractability of Un/Satisfiability

Latif Salum

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Latif Salum
Department of Industrial Engineering, Dokuz Eylül University, Izmir, Turkey
latif.salum@deu.edu.tr & latif.salum@gmail.com

Abstract

This paper shows $\mathbf{P} = \mathbf{NP}$ via exactly-1 3SAT (X3SAT). Let $\phi = \bigwedge C_k$ be some X3SAT formula. $C_k = (r_i \odot r_j \odot r_u)$ is a clause denoting an exactly-1 disjunction $\odot$ of literals r_i, $r_i \in \{x_i, \overline{x}_i\}$. C_k is satisfied iff $(r_i \wedge \overline{r}_j \wedge \overline{r}_u) \vee (\overline{r}_i \wedge r_j \wedge \overline{r}_u) \vee (\overline{r}_i \wedge \overline{r}_j \wedge r_u)$ is satisfied, because any C_k contains *exactly one* true literal by the definition of X3SAT. Let $\phi(r_j) := r_j \wedge \phi$. Then, r_j leads to reductions due to $\odot$ of some $C_k = (\overline{x}_i \odot r_j \odot x_u)$ into $c_k = x_i \wedge r_j \wedge \overline{x}_u$, and some $C_k = (\overline{r}_j \odot r_u \odot r_v)$ into $C_{k'} = (r_u \odot r_v)$. As a result, r_j transforms ϕ into $\phi(r_j) = \psi(r_j) \wedge \phi'(r_j)$, unless $\nvDash \psi(r_j)$, that is, unless $\psi(r_j)$ involves a contradiction $x_i \wedge \overline{x}_i$. Also, $\psi(r_j)$ and $\phi'(r_j)$ become *disjoint*, where $\psi(r_j) = \bigwedge(c_k \wedge C_{k'})$ for $|C_{k'}| = 1$, and $\phi'(r_j) = \bigwedge(C_k \wedge C_{k'})$. It is trivial to verify $\nvDash \psi(r_j)$ and *redundant* to verify $\nvDash \phi'(r_j)$, thus *easy* to verify $\nvDash \phi(r_j)$. A proof is sketched as follows. ϕ transforms into $\psi \wedge \phi'$ such that whenever $\nvDash \psi(r_j)$, $\overline{r}_j$ is placed in ψ, and leads to reductions of some C_k in ϕ'. If ψ involves $x_j \wedge \overline{x}_j$, then ϕ is unsatisfiable. Otherwise, ϕ is satisfiable, because ϕ is composed of $\psi, \psi(r_{i_0}), \psi(r_{i_1}|r_{i_0}), \ldots, \psi(r_{i_n}|r_{i_m})$, and all $\psi(.)$ are *disjoint* and *satisfied*. Note that $r_i \vDash \psi(r_i)$ and $\psi(r_i) \vDash \psi(r_i|.)$ for any r_i in ϕ'. Thus, $\phi'(r_i)$ is *satisfiable*, because $\phi \equiv \psi(r_i) \wedge \phi'(r_i)$, where $\psi(r_i)$ and $\phi'(r_i)$ are *disjoint*. Therefore, it is *redundant* to check if $\nvDash \phi'(r_i)$ to verify $\nvDash \phi(r_i)$, QED. The time complexity is $O(mn^3)$. Therefore, $\mathbf{P} = \mathbf{NP}$.

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1 Introduction: Effectiveness of X3SAT in proving $\mathrm{P} = \mathrm{NP}$

As is well known, $\mathbf{P} = \mathbf{NP}$, if there exists an efficient algorithm for any *one* of $\mathbf{NP}$-complete problems. That is, their algorithmic efficiency is *equivalent*. Nevertheless, some $\mathbf{NP}$-complete problem features algorithmic effectiveness, if it incorporates an *effective* tool to develop an efficient algorithm. That is, a particular problem can be more effective to prove $\mathbf{P} = \mathbf{NP}$. This issue might also be related to "complexity reductions" (Lipton and Regan [1]). They state these reductions are needed to understand what the $\mathbf{P} = \mathbf{NP}$ problem is really about.

The paper shows that one-in-three SAT, which is $\mathbf{NP}$-complete [3], features algorithmic effectiveness to prove $\mathbf{P} = \mathbf{NP}$. This problem is also known as exactly-1 3SAT (X3SAT). It incorporates "exactly-1 disjunction", denoted by $\odot$, the tool used to develop an efficient (or a polynomial time) algorithm, which "scans" an X3SAT formula ϕ, thus is called the ϕ scan.

If $\nvDash \phi(r_j)$, that is, $\phi(r_j)$ is unsatisfiable, then r_j is incompatible, where $\phi(r_j) := r_j \wedge \phi$ and $r_j \in \{x_j, \overline{x}_j\}$. The ϕ scan removes each incompatible r_j from ϕ, thus verifies compatibility of any r_i for satisfying ϕ. When each r_j incompatible is removed, ϕ is unsatisfiable, or satisfiable. If ϕ is satisfiable, then any r_i becomes compatible to participate in a satisfying assignment.

Let $\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ be an X3SAT formula, in which a clause $C_k = (r_i \odot r_j \odot r_u)$ is an exactly-1 disjunction of literals. C_k is satisfied by definition iff *exactly one* of r_i, r_j, or r_u is true. Note that $(r_i \vee r_j \vee r_u)$ in a 3SAT formula is satisfied iff at least one of them is true.

Incompatibility of r_i is checked by a *deterministic* chain of *reductions* of some C_k in $\phi(r_i)$. Consider $\phi(x_j) \coloneqq x_j \wedge \phi$. Then, the reductions are initiated by x_j, and followed by $\neg\overline{x}_j$, since $x_j \Rightarrow \neg\overline{x}_j$. That is, each $(x_j \odot \overline{x}_i \odot x_u)$ *collapses* to $(x_j \wedge x_i \wedge \overline{x}_u)$ due to $x_j \Rightarrow x_j \wedge \neg\overline{x}_i \wedge \neg x_u$, since there is exactly one (negated) variable that is true in any C_k by the definition of X3SAT. Also, each $(\overline{x}_j \odot \overline{x}_u \odot x_v)$ *shrinks* to $(\overline{x}_u \odot x_v)$ due to $\neg\overline{x}_j$. As a result, x_j transforms ϕ into $\phi(x_j) = x_j \wedge x_i \wedge \overline{x}_u \wedge \phi^*$, and $x_i \wedge \overline{x}_u$ proceeds the reductions in ϕ^*, which involves $(\overline{x}_u \odot x_v)$.

The reductions over $\phi_s(x_j)$ terminate iff x_j transforms ϕ_s into $\psi_s(x_j) \wedge \phi'_s(x_j)$, in which $\psi_s(x_j)$ and $\phi'_s(x_j)$ are disjoint, where s denotes the current scan, and $\psi_s(x_j)$ is a conjunction of (negated) variables that are true. They are interrupted iff $\psi_s(x_j)$ involves $x_i \wedge \overline{x}_i$, hence $\nvDash \phi_s(x_j)$, thus x_j is incompatible. Note that $\nvDash \phi_s(.)$ is verified *only* by $\nvDash \psi_s(.)$ (see Figure 1).

The reductions over ϕ terminate iff ϕ transforms into $\psi \wedge \phi'$, in which ψ and ϕ' are disjoint, where $\psi = \overline{x}_5 \wedge x_n \wedge \cdots \wedge \overline{x}_2$ (Figure 1). Then, ϕ is updated, that is, $\phi \leftarrow \phi'$. The ϕ_s scan is interrupted iff ψ_s involves $x_i \wedge \overline{x}_i$ for some s and i, thus $\nvDash \phi$, that is, ϕ is unsatisfiable.

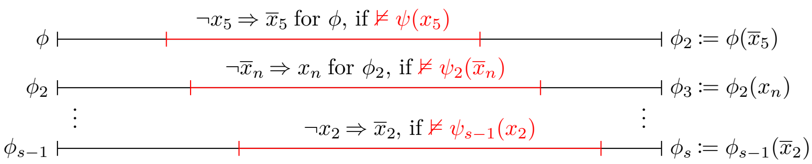

Figure 1 The ϕ_s scan: $\nvDash \phi_s(r_j)$ is verified *solely* by $\nvDash \psi_s(r_j)$ — whether or not $\nvDash \phi'_s(r_j)$ is *ignored*

▷ Claim 1. $\nvDash \phi(r_j)$ iff $\nvDash \psi_s(r_j)$ for some s. That is, it is *redundant* to check whether or not $\nvDash \phi'_s(r_j)$. Thus, $\phi(r_i)$ reduces to $\psi(r_i)$ due to $\phi(r_i) = \psi(r_i) \wedge \phi'(r_i)$. Then, $\psi(r_i) \equiv \phi(r_i)$. Therefore, ϕ is satisfiable iff $\psi(r_i)$ is *satisfied* for any r_i, that is, iff the ϕ_s scan *terminates*.

Sketch of proof. $\psi(r_i)/\psi(r_i|r_j)$ is constructed over $\phi/\phi'(r_j)$, thus $\psi(r_i)$ *covers* $\psi(r_i|r_j)$, hence $\psi(r_i) \vDash \psi(r_i|r_j)$ holds. Because $\psi(r_j)$ and $\phi'(r_j)$ are disjoint, $\psi(r_j)$ and $\psi(r_i|r_j)$ are disjoint (see Figure 2). Therefore, $\psi(r_{i_0})$, $\psi(r_{i_1}|r_{i_0})$, $\psi(r_{i_2}|r_{i_0}, r_{i_1})$, and $\psi(r_{i_3}|r_{i_0}, r_{i_1}, r_{i_2})$ form *disjoint* minterms $\psi(.) = \bigwedge r_i$ over ϕ such that $\psi(r_{i_0})$, $\psi(r_{i_1}|r_{i_0})$, $\psi(r_{i_2}|r_{i_0}, r_{i_1})$, and $\psi(r_{i_3}|r_{i_0}, r_{i_1}, r_{i_2})$ hold, since $\psi(r_i)$ is true for any r_i (the ϕ_s scan terminates), and $\psi(r_i) \vDash \psi(r_i|.)$ holds. Thus, ϕ is composed of $\psi(.)$ that are *disjoint* and *satisfied* (see Figure 3), hence ϕ is satisfied. ◁

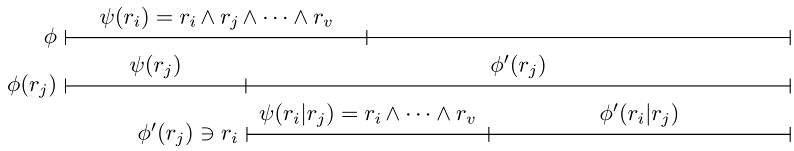

Figure 2 $\psi(r_i) \vDash \psi(r_i|r_j)$, and $\psi(r_j)$ and $\psi(r_i|r_j)$ are disjoint, thus $\psi(r_j) \wedge \psi(r_i|r_j)$ is true

A satisfying assignment α is constructed by composing $\psi(.)$ that are *disjoint* and *satisfied*. For example, $\alpha = \{\psi, \psi(r_{i_0}), \psi(r_{i_1}|r_{i_0}), \psi(r_{i_2}|r_{i_0}, r_{i_1}), \psi(r_{i_3}|r_{i_0}, r_{i_1}, r_{i_2})\}$ (see Figure 3).

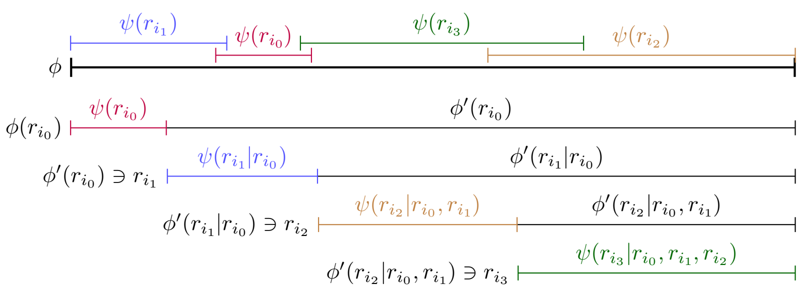

Figure 3 $\psi(r_{i_1}) \vDash \psi(r_{i_1}|r_{i_0})$, $\psi(r_{i_2}) \vDash \psi(r_{i_2}|r_{i_0}, r_{i_1})$, and $\psi(r_{i_3}) \vDash \psi(r_{i_3}|r_{i_0}, r_{i_1}, r_{i_2})$

2 Basic Definitions

This section gives basic definitions, which are based on exactly-1 disjunction, denoted by $\odot$.

▶ **Definition 2.** *A literal r_i is a variable x_i assigned true, or a negated variable $\overline{x}_i$ assigned true. That is, $r_i \in \{x_i, \overline{x}_i\}$, in which $x_i = \mathbf{T}$ and $\overline{x}_i = \mathbf{T}$.*

▶ **Definition 3.** *A clause $C_k = (r_i \odot r_j \odot r_u)$ denotes an exactly-1 disjunction of literals.*

▶ **Definition 4.** *$c_k = \bigwedge r_i$ denotes a* minterm, *a conjunction of r_i, where r_i is called a* conjunct.

▶ **Definition 5.** *$\varphi = \psi \wedge \phi$ denotes an* X3SAT *formula such that $\psi = \bigwedge c_k$ and $\phi = \bigwedge C_k$.*

Where appropriate, C_k, as well as ψ, is denoted by a set. Thus, $\varphi = \psi \wedge \phi$ the formula, that is, $\varphi = \psi \wedge C_1 \wedge C_2 \wedge \cdots \wedge C_m$, is denoted by $\varphi = \{\psi, C_1, C_2, \ldots, C_m\}$ the family of sets.

▶ **Definition 6.** *$C_k = (r_i \odot r_j \odot r_u)$ is satisfied iff $(r_i \wedge \overline{r}_j \wedge \overline{r}_u) \vee (\overline{r}_i \wedge r_j \wedge \overline{r}_u) \vee (\overline{r}_i \wedge \overline{r}_j \wedge r_u)$ is satisfied, since any clause C_k contains* exactly one *true literal by the definition of* X3SAT.

▶ **Definition 7** (Incompatibility). *r_i in some C_k is incompatible, denoted by $\neg r_i$, iff r_i leads to a contradiction $x_j \wedge \overline{x}_j$, that is, $r_i \wedge \varphi$ is unsatisfiable, hence r_i is removed from every C_k in ϕ.*

▶ Remark. Each x_i and $\overline{x}_i$ in ϕ is assumed to be compatible, thus no C_k contains $\neg x_i$, or $\neg \overline{x}_i$, while any r_i in ψ is *necessarily* true by Definition 4/5, thus denotes a *conjunct*, to satisfy φ.

▶ Note 8. If $r_i \in \psi$, then $r_i \Rightarrow \neg \overline{r}_i$, that is, $\overline{r}_i$ becomes incompatible, and is *removed* from ϕ. If $r_i \Rightarrow x_j \wedge \overline{x}_j$, hence $\neg x_j \vee \neg \overline{x}_j \Rightarrow \neg r_i$, then $\neg r_i \Rightarrow \overline{r}_i$, that is, $\overline{r}_i$ becomes a *conjunct* ($\overline{r}_i \in \psi$).

▶ **Definition 9.** *$\mathfrak{L} = \{1, 2, \ldots, n\}$ denotes the index set of the literals r_i, $\mathfrak{C} = \{1, 2, \ldots, m\}$ denotes the index set of the clauses C_k, and $\mathfrak{C}^{r_i} = \{k \in \mathfrak{C} \,|\, r_i \in C_k\}$ denotes C_k containing r_i.*

▶ **Example 10.** $\varphi = \overline{x}_4 \wedge (x_1 \odot \overline{x}_2 \odot x_3) \wedge (\overline{x}_3 \odot \overline{x}_4)$, in which $\overline{x}_4$ is necessary for satisfying φ, thus $\psi = \{\overline{x}_4\}$, $\mathfrak{C}^{\overline{x}_4} = \{2\}$, and $C_1 = \{x_1, \overline{x}_2, x_3\}$ denotes *either* $x_1 = \mathbf{T}$ *or* $\overline{x}_2 = \mathbf{T}$ *or* $x_3 = \mathbf{T}$.

▶ **Definition 11** (Collapse). *A clause $C_k = (r_i \odot x_j \odot \overline{x}_u)$ is said to* collapse *to the minterm $c_k = (r_i \wedge \overline{x}_j \wedge x_u)$, thus $r_i \notin C_k$, if r_i is* necessary, *denoted by $(r_i \odot x_j \odot \overline{x}_u) \searrow (r_i \wedge \overline{x}_j \wedge x_u)$.*

▶ **Definition 12** (Shrinkage). *A clause $C_k = (r_i \odot r_j \odot r_u)$ is said to shrink to another clause $C_{k'} = (r_j \odot r_u)$, if $\neg r_i$ (r_i* the incompatible is removed*), denoted by $(r_i \odot r_j \odot r_u) \rightarrowtail (r_j \odot r_u)$.*

▶ **Definition 13** (Compatibility of $r_i \in \{x_i, \overline{x}_i\}$ over ϕ). *$\phi(r_i) = r_i \wedge \phi$ for any $r_i \in C_k$ in ϕ.*

▶ Note 14 (Reduction). The collapse or shrinkage denotes a reduction. If $r_i \in \psi$, then r_i leads to *reductions* over ϕ, thus $\varphi \rightarrow \varphi'$. That is, $\varphi \rightarrow \varphi'$ iff $C_k \searrow c_k$ or $C_k \rightarrowtail C_{k'}$ for C_k in ϕ. Since r_i is necessary for $\phi(r_i)$, it leads to *reductions* over $\phi(r_i)$. Then, $(\overline{r}_i \odot r_v \odot r_y) \rightarrowtail (r_v \odot r_y)$ and $(r_i \odot x_j \odot \overline{x}_u) \searrow (r_i \wedge \overline{x}_j \wedge x_u)$, because $r_i \Rightarrow \neg \overline{r}_i$ such that $r_i \Rightarrow r_i \wedge \overline{x}_j \wedge x_u$ holds over some $C_k = (r_i \odot x_j \odot \overline{x}_u)$, since $r_i \Rightarrow \neg x_j \wedge \neg \overline{x}_u$, thus $\neg x_j \Rightarrow \overline{x}_j$ and $\neg \overline{x}_u \Rightarrow x_u$ (see Definition 6/7).

▶ **Definition 15.** *ϕ denotes a general formula if $\{x_i, \overline{x}_i\} \nsubseteq C_k$ for any $i \in \mathfrak{L}$ and $k \in \mathfrak{C}$, hence $\mathfrak{C}^{x_i} \cap \mathfrak{C}^{\overline{x}_i} = \emptyset$. ϕ denotes a* special *formula if $\{x_i, \overline{x}_i\} \subseteq C_k$ for some k, hence $\mathfrak{C}^{x_i} \cap \mathfrak{C}^{\overline{x}_i} = \{k\}$.*

▶ **Lemma 16** (Conversion of a special formula). *Each* clause *$C_k = (r_j \odot x_i \odot \overline{x}_i)$ is replaced by the* conjunct *$\overline{r}_j$ so that $\mathfrak{C}^{x_i} \cap \mathfrak{C}^{\overline{x}_i} = \emptyset$ for any $i \in \mathfrak{L}$, if $\phi = \bigwedge C_k$ is a special formula.*

Proof. ϕ is unsatisfiable due to $r_j \Rightarrow \overline{x}_i \wedge x_i$. Then, $x_i \vee \overline{x}_i \Rightarrow \overline{r}_j$. That is, $\overline{r}_j$ is *necessary* for satisfying $C_k = (r_j \odot x_i \odot \overline{x}_i)$, which is sufficient also, thus $\overline{r}_j$ is equivalent to C_k. Therefore, each *clause* $C_k = (r_j \odot x_i \odot \overline{x}_i)$ is replaced by the *conjunct* $\overline{r}_j$ so that $\mathfrak{C}^{x_i} \cap \mathfrak{C}^{\overline{x}_i} = \emptyset$. ◀

▶ **Example 17.** $\varphi = (x_2 \odot \overline{x}_1) \wedge (x_1 \odot \overline{x}_3 \odot x_4) \wedge (x_1 \odot \overline{x}_2 \odot x_2)$ is a special formula due to $C_3 = \{x_1, \overline{x}_2, x_2\}$. Note that $\mathfrak{C}^{\overline{x}_2} \cap \mathfrak{C}^{x_2} = \{3\}$. Then, φ is converted by replacing the clause C_3 with the conjunct $\overline{x}_1$. As a result, $\varphi \leftarrow \overline{x}_1 \wedge (x_2 \odot \overline{x}_1) \wedge (x_1 \odot \overline{x}_3 \odot x_4)$. Likewise, if $\varphi = (x_3 \odot \overline{x}_4 \odot x_4) \wedge (\overline{x}_3 \odot x_2 \odot \overline{x}_2) \wedge (x_2 \odot \overline{x}_1)$, then $\varphi \leftarrow \overline{x}_3 \wedge x_3 \wedge (x_2 \odot \overline{x}_1)$, which is unsatisfiable.

3 The φ Scan

The φ scan asserts that φ is satisfiable iff x_i or $\overline{x}_i$ is compatible (Definition 13) for all $i \in \mathfrak{L}$. Hence, we need to show that $\phi(x_1)$ or $\phi(\overline{x}_1)$, and $\phi(x_2)$ or $\phi(\overline{x}_2)$, and $\cdots$ and $\phi(x_n)$ or $\phi(\overline{x}_n)$ are satisfied. If φ is satisfiable, then a satisfying assignment is determined (see Section 3.4).

$\nvDash \varphi$ denotes φ is unsatisfiable, and $\vDash_\alpha \varphi$ denotes that $\alpha = \{r_1, r_2, \ldots, r_n\}$ is a satisfying assignment for φ. $\psi \vDash \psi'$ denotes that ψ *entails* ψ', and $\psi \vdash \psi'$ denotes that ψ *proves* ψ'.

φ_s for $s \geqslant 2$ denotes the *current* formula at the s^{th} scan/step such that $\varphi \coloneqq \varphi_1$, after $\neg r_j$ holds in ϕ_{s-1} (see Definition 7). Then, $\phi_s^{r_i} = (r_{ik_1} \odot r_{u_1k_1} \odot r_{u_2k_1}) \wedge \cdots \wedge (r_{ik_r} \odot r_{v_1k_r} \odot r_{v_2k_r})$ denotes the formula over clauses $C_k \ni r_i$ in ϕ_s, where $r_i \in \{x_i, \overline{x}_i\}$. Hence, $\mathfrak{C}_s^{r_i} = \{k_1, \ldots, k_r\}$.

$\tilde{\psi}_s(r_i)$ is called the *local* effect of r_i, and $\tilde{\phi}_s(\neg r_i)$ is the effect of $\neg r_i$. $\tilde{\varphi}_s(r_i)$ denotes its *overall* effect such that $\tilde{\varphi}_s(r_i) = \tilde{\psi}_s(r_i) \wedge \tilde{\phi}_s(\neg \overline{r}_i)$, specified below. Also, $\tilde{\psi}_s(r_i) = \bigwedge(c_k \wedge C_k)$ such that $|C_k| = 1$. Moreover, $\tilde{\phi}_s(\neg r_i) = \bigwedge C_k$ such that $|C_k| > 1$, or $\tilde{\phi}_s(\neg r_i)$ is empty.

3.1 Introduction: Incompatibility and Reductions

Example 18 (19) introduces incompatibility (reductions over ϕ), which drive the φ scan.

▶ **Example 18.** Consider $\phi(x_1)$ over $\varphi = \phi = (x_1 \odot \overline{x}_3) \wedge (x_1 \odot \overline{x}_2 \odot x_3) \wedge (x_2 \odot \overline{x}_3)$. Thus, x_1 is necessary for $\phi(x_1)$, hence $x_1 \vDash \tilde{\psi}(x_1)$ such that $\tilde{\psi}(x_1) = (x_1 \wedge x_3) \wedge (x_1 \wedge x_2 \wedge \overline{x}_3)$. That is, $x_1 \Rightarrow \neg\overline{x}_3$ holds over $C_1 = (x_1 \odot \overline{x}_3)$, hence $\neg\overline{x}_3 \Rightarrow x_3$. Likewise, $x_1 \Rightarrow \neg\overline{x}_2 \wedge \neg x_3$ holds over $(x_1 \odot \overline{x}_2 \odot x_3)$, hence $\neg\overline{x}_2 \Rightarrow x_2$ and $\neg x_3 \Rightarrow \overline{x}_3$ (see Note 14). Thus, $\tilde{\varphi}(x_1) = \tilde{\psi}(x_1) \wedge \tilde{\phi}(\neg\overline{x}_1)$ becomes the overall effect, where $\tilde{\phi}(\neg\overline{x}_1)$ is empty. Then, the reductions initiated by x_1 over $\phi(x_1)$ are to proceed due to x_2. Nevertheless, they are interrupted by $x_3 \wedge \overline{x}_3$ due to $\tilde{\psi}(x_1)$. Hence, $\phi(x_1) = \tilde{\varphi}(x_1) \wedge (x_2 \odot \overline{x}_3)$ is unsatisfiable, thus x_1 is *incompatible* for φ, i.e, $\neg x_1 \Rightarrow \overline{x}_1$.

▶ **Example 19.** $\overline{x}_1$ initiates *reductions* over ϕ (Note 14). Then, $\tilde{\psi}(\overline{x}_1) = \overline{x}_1 \wedge \overline{x}_3$, $\tilde{\phi}(\neg x_1) = (\overline{x}_2 \odot x_3)$, and $\tilde{\varphi}(\overline{x}_1) = \tilde{\psi}(\overline{x}_1) \wedge \tilde{\phi}(\neg x_1)$ to define $\varphi_2 = \tilde{\varphi}(\overline{x}_1) \wedge (x_2 \odot \overline{x}_3)$. Note that $(x_2 \odot \overline{x}_3)$ is beyond $\tilde{\varphi}(\overline{x}_1)$ the overall effect. Note also that $\{\overline{x}_3\} \notin \tilde{\phi}(\neg x_1)$, while $\overline{x}_3 \in \tilde{\psi}(\overline{x}_1)$, because $C_1 \rightarrowtail c_1$, since $\tilde{\phi}(\neg x_1)$ contains no singleton. Then, φ_2 is the current formula due to the first reduction by $\overline{x}_1$ over ϕ. Thus, $\varphi \rightarrow \varphi_2$ due to $(x_1 \odot \overline{x}_3) \rightarrowtail (\overline{x}_3)$ and $(x_1 \odot \overline{x}_2 \odot x_3) \rightarrowtail (\overline{x}_2 \odot x_3)$. As a result, $\varphi_2 = \overline{x}_1 \wedge \overline{x}_3 \wedge (\overline{x}_2 \odot x_3) \wedge (x_2 \odot \overline{x}_3)$, in which $\psi_2 = \{\overline{x}_1, \overline{x}_3\}$ denotes the conjuncts, and $C_1 = \{\overline{x}_2, x_3\}$ and $C_2 = \{x_2, \overline{x}_3\}$ denote the clauses. Note that $\mathfrak{C}_2^{x_3} = \{1\}$ and $\mathfrak{C}_2^{\overline{x}_3} = \{2\}$. Likewise, $\overline{x}_3$ leads to the next reduction over ϕ_2: $\tilde{\psi}_2(\overline{x}_3) = (\overline{x}_2 \wedge \overline{x}_3)$, $\tilde{\phi}_2(\neg x_3)$ is empty, and $\tilde{\varphi}_2(\overline{x}_3) = \tilde{\psi}_2(\overline{x}_3) \wedge \tilde{\phi}_2(\neg x_3)$. Thus, $\varphi_2 \rightarrow \varphi_3$ due to $(x_2 \odot \overline{x}_3) \searrow (\overline{x}_2 \wedge \overline{x}_3)$ and $(\overline{x}_2 \odot x_3) \rightarrowtail (\overline{x}_2)$. Then, $\varphi_3 = \tilde{\varphi}(\overline{x}_1) \wedge \tilde{\varphi}_2(\overline{x}_3) = \overline{x}_1 \wedge \overline{x}_2 \wedge \overline{x}_3$, which denotes the cumulative effects of $\overline{x}_1$ and $\overline{x}_3$.

3.2 The Core Algorithms: `Scope` and `Scan`

Let $\phi_s^{r_j} = (r_{jk_1} \odot r_{i_1k_1} \odot r_{i_2k_1}) \wedge \cdots \wedge (r_{jk_r} \odot r_{u_1k_r} \odot r_{u_2k_r})$ for Lemma 20 and 21 below.

▶ **Lemma 20.** *$r_j \vDash \tilde{\psi}_s(r_j)$ such that $\tilde{\psi}_s(r_j) = r_j \wedge \overline{r}_{i_1} \wedge \overline{r}_{i_2} \wedge \cdots \wedge \overline{r}_{u_1} \wedge \overline{r}_{u_2}$, unless $\nvDash \tilde{\psi}_s(r_j)$.*

Proof. Follows from Definition 11. That is, $r_j \Rightarrow (r_j \wedge \overline{r}_{i_1} \wedge \overline{r}_{i_2}) \wedge \cdots \wedge (r_j \wedge \overline{r}_{u_1} \wedge \overline{r}_{u_2})$. Hence, $r_j \Rightarrow r_j \wedge \overline{r}_{i_1} \wedge \overline{r}_{i_2} \wedge \cdots \wedge \overline{r}_{u_1} \wedge \overline{r}_{u_2}$. ◀

▶ **Lemma 21.** *If $\neg r_j$, then $\tilde{\phi}_s(\neg r_j)$ holds such that $\tilde{\phi}_s(\neg r_j) = (r_{i_1} \odot r_{i_2}) \wedge \cdots \wedge (r_{u_1} \odot r_{u_2})$.*

Proof. Follows from Definition 12. $\tilde{\phi}_s(\neg r_j) = \{\{\}\}$, or $|C_k| > 1$ for any C_k in $\tilde{\phi}_s(\neg r_j)$. ◀

▶ **Lemma 22** (Overall effect of r_j over ϕ_s)**.** $\tilde{\varphi}_s(r_j) = \tilde{\psi}_s(r_j) \wedge \tilde{\phi}_s(\neg \overline{r}_j)$.

Proof. Follows from Lemma 20, and from 21 via $\phi_s^{\overline{r}_j}$, since $r_j \Rightarrow \neg\overline{r}_j$, thus $r_j \vDash r_j \wedge \neg\overline{r}_j$. ◀

The algorithm $\texttt{OvrlEft}\,(r_j, \phi_*)$ below constructs the overall effect $\tilde{\varphi}_*(r_j)$ by means of the local effect $\tilde{\psi}_*(r_j)$ (see Lines 1-6, or L:1-6), as well as of the local effect $\tilde{\phi}_*(\neg\overline{r}_j)$ (L:7-10).

Algorithm 1 $\texttt{OvrlEft}\,(r_j, \phi_*)$ ▷ Construction of the overall effect $\tilde{\varphi}_*(r_j)$ due to Lemma 22

1: **for all** $k \in \mathfrak{C}_*^{r_j}$ over ϕ_* **do** ▷ Construction of the local effect $\tilde{\psi}_*(r_j)$ due to r_j (Lemma 20)
2: **for all** $r_i \in \big(C_k - \{r_j\}\big)$ **do** ▷ $\tilde{\psi}_*(r_j)$ gets r_j via r_e (see Scope L:4), or via $\overline{r}_j$ (Remove L:2)
3: $c_k \leftarrow c_k \cup \{\overline{r}_i\}$; ▷ $(r_{jk} \odot r_{i_1k} \odot r_{i_2k}) \searrow (\overline{r}_{i_1k} \wedge \overline{r}_{i_2k})$. That is, $C_k \searrow c_k$ (see Definition 4/11)
4: **end for**
5: $\tilde{\psi}_*(r_j) \leftarrow \tilde{\psi}_*(r_j) \cup c_k$; ▷ c_k consists in $\psi_s(r_j)$ (see Scope L:4), or in ψ_s (see Remove L:2)
6: **end for** ▷ L:1-6 are independent from L:7-10, since $\mathfrak{C}_*^{r_j} \cap \mathfrak{C}_*^{\overline{r}_j} = \emptyset$, i.e., $\mathfrak{C}_*^{x_j} \cap \mathfrak{C}_*^{\overline{x}_j} = \emptyset$ (Lemma 16)
7: **for all** $k \in \mathfrak{C}_*^{\overline{r}_j}$ over ϕ_* **do** ▷ Construction of the local effect $\tilde{\phi}_*(\neg\overline{r}_j)$ due to $\neg\overline{r}_j$ (Lemma 21)
8: $C_k \leftarrow C_k - \{\overline{r}_j\}$; ▷ $(\overline{r}_{jk} \odot r_{u_1k} \odot r_{u_2k}) \rightarrowtail (r_{u_1k} \odot r_{u_2k})$ or $(\overline{r}_{jk} \odot r_{uk}) \rightarrowtail (r_{uk})$ (Definition 12)
9: **if** $|C_k| = 1$ **then** $\tilde{\psi}_*(r_j) \leftarrow \tilde{\psi}_*(r_j) \cup C_k$; $C_k \leftarrow \emptyset$; ▷ $\tilde{\phi}_*(\neg\overline{r}_j)$ contains no singleton, $C_k \rightarrowtail c_k$
10: **end for** ▷ 3\2-literal C_k in $\phi_*^{\overline{r}_j}$ shrinks due to $\neg\overline{r}_j$ to 2-literal C_k in $\phi_*^{\overline{r}_j}$\to *conjunct* r_u in $\tilde{\psi}_*(r_j)$
11: **return** $\tilde{\psi}_*(r_j)$ & $\tilde{\phi}_*(\neg\overline{r}_j) \leftarrow \phi_*^{\overline{r}_j}$; ▷ $\tilde{\psi}_*(r_j) = \bigwedge(c_k \wedge C_k)$, $|C_k| = 1$ & $\tilde{\phi}_*(\neg\overline{r}_j) = \bigwedge C_k$, $|C_k| > 1$

$\psi_s(r_j)$ is called the scope of r_j, and $\phi_s'(r_j)$ is called beyond the scope, defined over ϕ_s.

▶ **Lemma 23** (Scope of r_j). *r_j transforms ϕ_s into $\phi_s(r_j) = \psi_s(r_j) \wedge \phi_s'(r_j)$, unless $\nvDash \psi_s(r_j)$, where $\psi_s(r_j) = r_j \wedge r_i \wedge \cdots \wedge r_u$ and $\phi_s'(r_j) = \bigwedge C_k$. Thus, $r_j \vDash \psi_s(r_j)$, hence $r_j \vdash \psi_s(r_j)$.*

Proof. $\phi_s(r_j) = r_j \wedge \phi_s$ by Definition 13. Then, r_j initiates a *deterministic* chain of reductions (see Note 14). As a result, $r_j \Rightarrow r_j \wedge x_i \wedge \overline{x}_u$ holds over each $C_k = (r_j \odot \overline{x}_i \odot x_u)$ containing r_j, and $\neg\overline{r}_j \Rightarrow (\overline{x}_u \odot x_v)$ holds over each $C_k = (\overline{r}_j \odot \overline{x}_u \odot x_v)$ containing $\overline{r}_j$. These reductions thus proceed, as long as new conjuncts r_e emerge in $\phi_s(r_j)$ (see Scope L:2-4). If the reductions are interrupted, then r_j is incompatible (L:5). If they terminate, then $\psi_s(r_j)$ and $\phi_s'(r_j)$ are constructed (L:9). Thus, $r_j \vDash \psi_s(r_j)$. It is obvious that if $r_j \vDash \psi_s(r_j)$, then $r_j \vdash \psi_s(r_j)$. ◀

Algorithm 2 $\texttt{Scope}\,(r_j, \phi_s)$ ▷ Construction of $\psi_s(r_j)$ and $\phi_s'(r_j)$ due to r_j over ϕ_s; $\varphi_s = \psi_s \wedge \phi_s$

1: $\psi_s(r_j) \leftarrow \{r_j\}$; $\phi_* \leftarrow \phi_s$; ▷ $\phi_s(r_j) \coloneqq r_j \wedge \phi_s$. ψ_s and ϕ_s are disjoint due to Scan L:1-3
2: **for all** $r_e \in \big(\psi_s(r_j) - R\big)$ **do** ▷ Reductions of C_k initiated by r_j over ϕ_s start off
3: $\texttt{OvrlEft}\,(r_e, \phi_*)$; ▷ It returns $\tilde{\psi}_*(r_e)$ for L:4 & $\tilde{\phi}_*(\neg\overline{r}_e)$ for L:6
4: $\psi_s(r_j) \leftarrow \psi_s(r_j) \cup \{r_e\} \cup \tilde{\psi}_*(r_e)$; ▷ $\tilde{\psi}_*(r_e)$ due to OvrlEft L:5,9 consists in the scope $\psi_s(r_j)$
5: **if** $\psi_s(r_j) \supseteq \{x_i, \overline{x}_i\}$ **then return** NULL; ▷ $r_j \Rightarrow x_i \wedge \overline{x}_i$, $i \in \mathfrak{L}^\phi$. $\nvDash \psi_s(r_j)$, thus $\nvDash \phi_s(r_j)$
6: $\tilde{\phi}_*(\neg r) \leftarrow \tilde{\phi}_*(\neg r) \cup \tilde{\phi}_*(\neg\overline{r}_e)$; ▷ $\tilde{\phi}_*(\neg r) = \big\{\{\}\big\}$ or $\tilde{\phi}_*(\neg r) = \bigcup C_k$, $|C_k| > 1$ (OvrlEft L:8-11)
7: $\phi_* \leftarrow \tilde{\phi}_*(\neg r) \wedge \phi_*'$; $R \leftarrow R \cup \{r_e\}$; ▷ $\tilde{\phi}_*(\neg r)$ and ϕ_*' consist in beyond the scope $\phi_s'(r_j)$
▷ $\phi_*' = \bigwedge C_k$ for $k \in \mathfrak{C}_*'$, where $\mathfrak{C}_*' = \mathfrak{C}_* - (\mathfrak{C}_*^{x_e} \cup \mathfrak{C}_*^{\overline{x}_e})$, and $\mathfrak{C}_*^{x_e} \cap \mathfrak{C}_*^{\overline{x}_e} = \emptyset$ due to Lemma 16
8: **end for** ▷ The reductions terminate if $\psi_s(r_j) = R$, which denotes conjuncts already reduced C_k
9: **return** $\psi_s(r_j)$ & $\phi_s'(r_j) \leftarrow \phi_*$; ▷ $\phi_s(r_j) = \psi_s(r_j) \wedge \phi_s'(r_j)$. $\psi_s(r_j) = \bigwedge r_j$ and $\phi_s'(r_j) = \bigwedge C_k$

▶ Note 24. $\mathfrak{L}_s(r_j)$ being an index set of $\psi_s(r_j)$, $\mathfrak{L}_s(r_j) \cap \mathfrak{L}_s'(r_j) = \emptyset$ and $\mathfrak{L}_s(r_j) \cup \mathfrak{L}_s'(r_j) = \mathfrak{L}^\phi$, if $\texttt{Scope}\,(r_j, \phi_s)$ terminates. Thus, $\psi_s(r_j)$ and $\phi_s'(r_j)$ are disjoint, where $\phi_s'(r_j)$ can be empty.

▶ **Example 25.** Consider $\psi(x_1)$, $\texttt{Scope}\,(x_1, \phi)$, for $\phi = (x_1 \odot \overline{x}_3) \wedge (x_1 \odot \overline{x}_2 \odot x_3) \wedge (x_2 \odot \overline{x}_3)$. $\psi(x_1) \leftarrow \{x_1\}$ and $\phi_* \leftarrow \phi$ (L:1). Then, $\phi_*^{\overline{x}_1}$ is empty, and $\phi_*^{x_1} = (x_1 \odot \overline{x}_3) \wedge (x_1 \odot \overline{x}_2 \odot x_3)$ due to $\texttt{OvrlEft}\,(x_1, \phi_*)$. Also, $\mathfrak{C}_*^{x_1} = \{1, 2\}$, thus $c_1 \leftarrow \{x_3\}$ and $\tilde{\psi}_*(x_1) \leftarrow \tilde{\psi}_*(x_1) \cup c_1$, as well as $c_2 \leftarrow \{x_2, \overline{x}_3\}$ and $\tilde{\psi}_*(x_1) \leftarrow \tilde{\psi}_*(x_1) \cup c_2$ (see OvrlEft L:1-6). Then, $\tilde{\psi}_*(x_1) = \{x_3, x_2, \overline{x}_3\}$ & $\tilde{\phi}_*(\neg\overline{x}_1) \leftarrow \phi_*^{\overline{x}_1}$ (OvrlEft L:11). As a result, $\psi(x_1) \leftarrow \psi(x_1) \cup \{x_1\} \cup \tilde{\psi}_*(x_1)$ (Scope L:4), and $\psi(x_1) \supseteq \{x_3, \overline{x}_3\}$ (L:5), that is, $x_1 \Rightarrow x_3 \wedge \overline{x}_3$, hence x_1 is incompatible in the *first* scan.

▶ **Definition 26.** $\mathfrak{L}^{\psi} = \{i \in \mathfrak{L} \,|\, r_i \in \psi_s\}$ *and* $\mathfrak{L}^{\phi} = \{i \in \mathfrak{L} \,|\, r_i \in C_k$ *in* $\phi_s\}$ *due to* $\varphi_s = \psi_s \wedge \phi_s$.

`Scan` (φ_s) decomposes ϕ_s into $\psi_s(x_1), \psi_s(\overline{x}_1), \ldots, \psi_s(x_n), \psi_s(\overline{x}_n)$, *whenever* $\mathfrak{L}^{\psi} \cap \mathfrak{L}^{\phi} = \emptyset$. If $\nvDash \psi_{s-1}(r_i)$, then $\overline{r}_i$ is placed in ψ_s, and leads to reductions of some C_k in ϕ_s. In Figure 4, $\nvDash \psi_{s-2}(\overline{x}_1)$ and $\nvDash \psi_{s-1}(x_3)$ hold, thus $\psi_s = x_1 \wedge \overline{x}_3$ and $\phi_s = (x_4 \odot \overline{x}_2 \odot x_n) \wedge \cdots \wedge (x_2 \odot \overline{x}_n)$.

$$\varphi_s = \underbrace{x_1 \wedge \overline{x}_3}_{\psi_s} \wedge \underbrace{\underbrace{(x_4 \odot \overline{x}_2 \odot x_n)}_{C_1} \wedge \cdots \wedge \overbrace{(\overline{x}_6 \odot x_8) \wedge (\overline{x}_6 \odot \overline{x}_9 \odot x_4) \wedge (x_7 \odot x_8)}^{\psi_s(\overline{x}_6) = \overline{x}_6 \wedge \overline{x}_8 \wedge x_9 \wedge \overline{x}_4 \wedge x_7} \wedge \cdots \wedge \underbrace{(x_2 \odot \overline{x}_n)}_{C_m}}_{\phi_s}$$

Figure 4 `Scan` (φ_s) decomposes ϕ_s into $\psi_s(x_1), \psi_s(\overline{x}_1), \ldots, \psi_s(x_n), \psi_s(\overline{x}_n)$, unless $\psi_s(.) \not\supseteq \{x_i, \overline{x}_i\}$

If $\overline{r}_i \in \psi_s$, then $\overline{r}_i$ is necessary, thus r_i is incompatible *trivially* for each $C_k \ni r_i$ in ϕ_s (see `Scan` L:1-2). For example, if $x_1 \wedge (x_1 \odot x_2 \odot \overline{x}_3)$ holds, then $\overline{x}_1$ becomes incompatible trivially. Note that $1 \in \mathfrak{L}^{\phi}$ and $x_1 \in \psi_s$, and that $\overline{x}_1 \Rightarrow \overline{x}_1 \wedge x_1$. If $r_i \Rightarrow x_j \wedge \overline{x}_j$, then r_i is incompatible *nontrivially* (L:6). See also Note 8/27. If `Scan` (φ_s) is interrupted by `Remove` L:3, then φ is unsatisfiable. If it terminates (L:9), then a satisfying assignment is determined (Section 3.4).

▶ Note 27. It is obvious that $\nvDash \varphi_s(r_j)$ if $\nvDash (\psi_s \wedge r_j)$ or $\nvDash (r_j \wedge \phi_s)$ by Definition 5/13, because $\varphi_s(r_j) = \psi_s \wedge r_j \wedge \phi_s$, and $r_j \wedge \phi_s = \phi_s(r_j)$, and that $\nvDash \varphi_s(r_j)$ iff $\neg r_j$ holds (see Definition 7).

Algorithm 3 `Scan` (φ_s) ▷ $\varphi_s = \psi_s \wedge \phi_s$, $\psi_s = \bigwedge r_i$ and $\phi_s = \bigwedge C_k$. Checks if $\nvDash \varphi_s(r_i)$ for all $i \in \mathfrak{L}^{\phi}$

1: **for all** $i \in \mathfrak{L}^{\phi}$ and $\overline{r}_i \in \psi_s$ **do** ▷ Because $\overline{r}_i \in \psi_s$, $\nvDash (\psi_s \wedge r_i)$, that is, $r_i \Rightarrow x_i \wedge \overline{x}_i$
2: `Remove` (r_i, ϕ_s); ▷ $\overline{r}_i$ is *necessary*, thus r_i is incompatible *trivially*, hence $\overline{r}_i \Rightarrow \neg r_i$
3: **end for** ▷ If $i \in \mathfrak{L}^{\psi}$, r_i has been already removed, hence $\overline{r}_i \in \psi_s$ and $\overline{r}_i \notin C_k \forall k \in \mathfrak{C}_s$, i.e., $i \notin \mathfrak{L}^{\phi}$
4: **for all** $i \in \mathfrak{L}^{\phi}$ **do** ▷ $\mathfrak{L}^{\psi} \cap \mathfrak{L}^{\phi} = \emptyset$ due to L:1-3. Hence, $i \in \mathfrak{L}^{\psi}$ iff $r_i = x_i$ is *fixed* or $r_i = \overline{x}_i$ is *fixed*
5: **for all** $r_i \in \{x_i, \overline{x}_i\}$ **do** ▷ Each and every x_i and $\overline{x}_i$ *assumed* compatible is to be *verified*
6: **if** `Scope` (r_i, ϕ_s) is NULL **then** `Remove` (r_i, ϕ_s); ▷ $\nvDash \phi_s(r_i)$, incompatible *nontrivially*
7: **end for** ▷ If $r_i \Rightarrow x_j \wedge \overline{x}_j$, hence $\neg x_j \vee \neg \overline{x}_j \Rightarrow \neg r_i$, then $\neg r_i \Rightarrow \overline{r}_i$, where $i \neq j$ due to L:1-3
8: **end for** ▷ $\neg r_i$ iff $\overline{r}_i$, since $\neg r_i \Rightarrow \overline{r}_i$ due to nontrivial, and $\neg r_i \Leftarrow \overline{r}_i$ due to trivial incompatibility
9: **return** $\hat{\varphi} = \hat{\psi} \wedge \hat{\phi}$, and $\psi(r_i)$ & $\phi'(r_i)$ for all $i \in \mathfrak{L}^{\hat{\phi}}$; ▷ $\hat{\psi} \leftarrow \psi_{\hat{s}}$ and $\hat{\phi} \leftarrow \phi_{\hat{s}}$. See also Note 29

▶ Note 28. $\mathfrak{L}^{\psi}$ and $\mathfrak{L}^{\phi}$ form a partition of $\mathfrak{L}$ due to Definition 26 and `Scan` L:1-3.

▶ Note 29. When `Scan` terminates, $\hat{\psi}$ and $\hat{\phi}$ become disjoint, and $\hat{\phi} \equiv \bigwedge_{i \in \mathfrak{L}} (\psi(x_i) \oplus \psi(\overline{x}_i))$, where $\mathfrak{L} \leftarrow \mathfrak{L}^{\hat{\phi}}$. Also, $\hat{\psi} = \bigwedge r_i$ and $\hat{\phi} = \bigwedge C_k$ such that $|C_k| > 1$, because each $C_k = \{r_i\}$ in ϕ_s for any s transforms into r_i in $\hat{\psi}$. That is, $C_k = (r_i \odot r_j)$ or $C_k = (r_i \odot r_j \odot r_u)$ in $\hat{\phi}$.

`Remove` (r_j, ϕ_s) leads to reductions of any $C_k \ni \overline{r}_j$ due to $\overline{r}_j$, which consists in ψ_{s+1} (see L:1-2), as well as of any $C_k \ni r_j$ due to $\neg r_j$, which consists in ϕ_{s+1} (see L:1,5).

Algorithm 4 `Remove` (r_j, ϕ_s) ▷ r_j is incompatible/removed iff $\overline{r}_j$ is necessary, i.e., $\neg r_j$ iff $\overline{r}_j$

1: `OvrlEft` $(\overline{r}_j, \phi_s)$; ▷ `OvrlEft` is defined over $\phi_s = \bigwedge C_k$, $|C_k| > 1$, and returns $\tilde{\psi}_s(\overline{r}_j)$ & $\tilde{\phi}_s(\neg r_j)$
2: $\psi_{s+1} \leftarrow \psi_s \cup \{\overline{r}_j\} \cup \tilde{\psi}_s(\overline{r}_j)$; ▷ $\psi_{s+1} = \bigwedge r_i$ is true by definition, unless ψ_{s+1} involves $x_i \wedge \overline{x}_i$
3: **if** $\psi_{s+1} \supseteq \{x_i, \overline{x}_i\}$ for some i **then return** φ is unsatisfiable; ▷ $\varphi_s = \psi_s \wedge \phi_s$
4: $\mathfrak{L}^{\phi} \leftarrow \mathfrak{L}^{\phi} - \{j\}$; $\mathfrak{L}^{\psi} \leftarrow \mathfrak{L}^{\psi} \cup \{j\}$;
5: $\phi_{s+1} \leftarrow \tilde{\phi}_s(\neg r_j) \wedge \phi'_s$; Update $\{C_k\}$ over ϕ_{s+1}; ▷ ϕ'_s denotes clauses beyond the entire ψ_s effect
▷ $\phi'_s = \bigwedge C_k$ for $k \in \mathfrak{C}'_s$, where $\mathfrak{C}'_s = \mathfrak{C}_s - (\mathfrak{C}_s^{\overline{x}_j} \cup \mathfrak{C}_s^{x_j})$, and $\mathfrak{C}_s^{\overline{x}_j} \cap \mathfrak{C}_s^{x_j} = \emptyset$ due to Lemma 16
6: `Scan` (φ_{s+1}); ▷ r_i verified compatible for $\check{s} \leqslant s$ can be incompatible for $\tilde{s} > s$ due to $\neg r_j$ in ϕ_s

3.3 Satisfiability of the Formula φ vs Satisfiability of the Scope $\psi(r_i)$

This section shows that φ is satisfiable iff $\psi(r_i)$ is satisfied for all $i \in \mathfrak{L}$, and any $r_i \in \{x_i, \overline{x}_i\}$.

▶ **Proposition 30** (Nontrivial incompatibility). *$\nvDash \phi_s(r_j)$ iff $\nvDash \psi_s(r_j)$ or $\nvDash \phi'_s(r_j)$ for any s.*

Proof. Proof is obvious due to $\phi_s(r_j) = \psi_s(r_j) \wedge \phi'_s(r_j)$ by Lemma 23. ◀

▶ Note 31 (Assumption). $\nvDash \phi_s(r_j)$ is verified *solely* via $\nvDash \psi_s(r_j)$ for some s, whether or not $\nvDash \phi'_s(r_j)$ is *ignored*, which is sufficient for incompatibility, and easy to check (see `Scope` L:5).

The following introduces the tools to justify this assumption, which facilitates the φ scan. Assume that `Scan` *terminates* (L:9), that is, $\psi \wedge \phi$ transforms into $\hat{\psi} \wedge \hat{\phi}$. Let $\phi \leftarrow \hat{\phi}$, thus $\mathfrak{L} \leftarrow \mathfrak{L}^{\hat{\phi}}$. Therefore, $r_i \vDash \psi(r_i)$ for all $i \in \mathfrak{L}$ and $r_i \in \{x_i, \overline{x}_i\}$. That is, as $r_i = \mathbf{T}$, $\psi(r_i) = \mathbf{T}$.

▶ **Definition 32.** *$\mathfrak{L}(.) = \mathfrak{L}(\psi(.))$ and $\mathfrak{L}'(.) = \mathfrak{L}(\phi'(.))$, which denote respective index sets.*

▶ **Lemma 33** (No conjunct exists in beyond the scope). *$\mathfrak{L}(r_j) \cap \mathfrak{L}'(r_j) = \emptyset$ for any $j \in \mathfrak{L}$.*

Proof. $\phi'(r_j) = \bigwedge C_k$ due to Lemma 23. Let r_i the *conjunct* be in C_k, i.e., $i \in \big(\mathfrak{L}(r_j) \cap \mathfrak{L}'(r_j)\big)$. Then, for any $C_k \ni r_i$, $(r_i \odot x_j \odot \overline{x}_u) \searrow (r_i \wedge \overline{x}_j \wedge x_u)$, thus $r_i \notin C_k$. Moreover, for any $C_k \ni \overline{r}_i$, $(\overline{r}_i \odot r_v \odot r_y) \rightarrowtail (r_v \odot r_y)$, thus $\overline{r}_i \notin C_k$. See Definition 11/12. Hence, $i \notin \big(\mathfrak{L}(r_j) \cap \mathfrak{L}'(r_j)\big)$. ◀

$\psi(r_i|r_j)$ is called the conditional scope, and $\phi'(r_i|r_j)$ is called conditional beyond the scope, which are defined over $\phi'(r_j)$ for $j \neq i$, that is, constructed by `Scope` $\big(r_i, \phi'(r_j)\big)$.

▶ **Lemma 34.** *$\mathfrak{L}$ is partitioned into $\mathfrak{L}(r_j)$, $\mathfrak{L}(r_{j_1}|r_j)$, $\mathfrak{L}(r_{j_2}|r_{j_1}), \ldots, \mathfrak{L}(r_{j_n}|r_{j_m})$, thus $\phi(r_j)$ is decomposed into* disjoint $\psi(r_j)$, $\psi(r_{j_1}|r_j)$, $\psi(r_{j_2}|r_{j_1}), \ldots, \psi(r_{j_n}|r_{j_m})$.

Proof. `Scope` (r_j, ϕ) *partitions* $\mathfrak{L}$ into $\mathfrak{L}(r_j)$ and $\mathfrak{L}'(r_j)$ for any $j \in \mathfrak{L}$ (see also Lemma 33). Thus, $\phi(r_j)$ is decomposed into *disjoint* $\psi(r_j)$ and $\phi'(r_j)$. Then, `Scope` $\big(r_{j_1}, \phi'(r_j)\big)$ *partitions* $\mathfrak{L}'(r_j)$ into $\mathfrak{L}(r_{j_1}|r_j)$ and $\mathfrak{L}'(r_{j_1}|r_j)$ for any $j_1 \in \mathfrak{L}'(r_j)$. Thus, $\phi'(r_j)$ is decomposed into *disjoint* $\psi(r_{j_1}|r_j)$ and $\phi'(r_{j_1}|r_j)$. Finally, $\phi'(r_{j_m}|r_{j_l})$ is decomposed into *disjoint* $\psi(r_{j_n}|r_{j_m})$ and $\phi'(r_{j_n}|r_{j_m})$ for any $j_n \in \mathfrak{L}'(r_{j_m}|r_{j_l})$ such that $\mathfrak{L}'(r_{j_n}|r_{j_m}) = \emptyset$ (see also Note 24). ◀

▶ **Lemma 35.** *$\phi'(r_j)$ is decomposed into* disjoint $\psi(r_{j_1}|r_j)$, $\psi(r_{j_2}|r_{j_1}), \ldots, \psi(r_{j_n}|r_{j_m})$.

Proof. Follows directly from Lemma 34, and from Lemma 23, $\phi(r_j) = \psi(r_j) \wedge \phi'(r_j)$. ◀

▶ **Lemma 36.** *$\phi \supseteq \phi'(r_j) \supseteq \phi'(r_{j_1}|r_j) \supseteq \phi'(r_{j_2}|r_{j_1}) \supseteq \cdots \supseteq \phi'(r_{j_m}|r_{j_l})$, when it* terminates.

Proof. Follows directly from Lemma 34. Then, some C_k in ϕ collapse to some c_k in $\psi(r_j)$. Thus, the number of C_k in ϕ is greater than or equal to that of C_k in $\phi'(r_j)$, hence $|\mathfrak{C}| \geqslant |\mathfrak{C}'|$, where $\mathfrak{C}$ is an index set of C_k in ϕ. Also, some C_k in ϕ shrink to some $C_{k'}$ in $\phi'(r_j)$, hence $\forall k' \in \mathfrak{C}' \, \exists k \in \mathfrak{C} \, [C_k \supseteq C_{k'}]$. Thus, $\phi \supseteq \phi'(r_j)$. Likewise, $\phi'(r_j) \supseteq \phi'(r_{j_1}|r_j)$, because $\phi'(r_j)$ is decomposed into $\psi(r_{j_1}|r_j)$ and $\phi'(r_{j_1}|r_j)$. Therefore, $\phi \supseteq \phi'(r_j) \supseteq \phi'(r_{j_1}|r_j) \supseteq \phi'(r_{j_2}|r_{j_1}) \supseteq \cdots \supseteq \phi'(r_{j_m}|r_{j_l})$, where $\phi'(r_{j_m}|r_{j_l}) = \phi'(r_{j_m}|r_j, \ldots, r_{j_l})$. Note that $\phi'(r_{j_n}|r_{j_m}) = \big\{\{\}\big\}$. ◀

▶ **Lemma 37.** *$\psi(r_i) \vDash \psi(r_i|r_j)$, thus $\psi(r_i) \vdash \psi(r_i|r_j)$, when the scan* terminates.

Proof. `Scope` (r_i, ϕ) constructs $\psi(r_i)$ and `Scope` $\big(r_i, \phi'(r_j)\big)$ constructs $\psi(r_i|r_j)$. $\phi \supseteq \phi'(r_j)$ by Lemma 36. Therefore, $\psi(r_i) \supseteq \psi(r_i|r_j)$, and $\psi(r_i) \vDash \psi(r_i|r_j)$ (see also Figure 2), where $\psi(r_i) = r_i \wedge r_j \wedge \cdots \wedge r_v$ and $\psi(r_i|r_j) = r_i \wedge \cdots \wedge r_v$. Then, $r_j \notin \psi(r_i|r_j)$, since $r_j \notin C_k$ for any $C_k \in \phi'(r_j)$ by Lemma 33. It is obvious that if $\psi(r_i) \vDash \psi(r_i|r_j)$, then $\psi(r_i) \vdash \psi(r_i|r_j)$. ◀

Lemma 37 leads to Lemma 38, because $r_i \vDash \psi(r_i)$ and $r_i \vdash \psi(r_i)$ by Lemma 23. That is, each and every conditional scope $\psi(r_i|.)$ is entailed and proved, when the scan *terminates*.

▶ **Lemma 38.** *$\psi(r_i|r_j)$, $\psi(r_i|r_j,r_{j_1}),\ldots,\psi(r_i|r_j,r_{j_1},\ldots,r_{j_m})$ holds for every $j\in\mathfrak{L}$, and for every $i\in\mathfrak{L}'(r_j)$, $i\in\mathfrak{L}'(r_{j_1}|r_j),\ldots,i\in\mathfrak{L}'(r_{j_m}|r_j,r_{j_1},\ldots,r_{j_l})$, when the scan* terminates.

Proof. $\phi\supseteq\phi'(r_j)\supseteq\phi'(r_{j_1}|r_j)\supseteq\cdots\supseteq\phi'(r_{j_m}|r_{j_l})$ by Lemma 36. Hence, $\psi(r_i)\supseteq\psi(r_i|r_j)$, $\psi(r_i)\supseteq\psi(r_i|r_j,r_{j_1}),\ldots,\psi(r_i)\supseteq\psi(r_i|r_j,\ldots,r_{j_m})$, and $\psi(r_i)\vDash\psi(r_i|r_j),\psi(r_i)\vDash\psi(r_i|r_j,r_{j_1})$, $\ldots,\psi(r_i)\vDash\psi(r_i|r_j,r_{j_1},\ldots,r_{j_m})$. Note that if $\psi(r_i)\vDash\psi(r_i|.)$, then $\psi(r_i)\vdash\psi(r_i|.)$. Therefore, $\psi(r_i|r_j)$, $\psi(r_i|r_j,r_{j_1}),\ldots,\psi(r_i|r_j,r_{j_1},\ldots,r_{j_m})$ hold, which generalizes Lemma 37. ◀

▶ **Theorem 39** (Unsatisfiability)**.** *r_j is incompatible due to $\nvDash\phi(r_j)$ iff $\nvDash\psi_s(r_j)$ for some s.*

▶ **Corollary 40** (Satisfiability)**.** *$\vDash_\alpha\phi$ iff the scope $\psi(r_i)$ holds for every $i\in\mathfrak{L}$ and $r_i\in\{x_i,\overline{x}_i\}$.*

Proof. $\psi(r_{j_1}|r_j)$, $\psi(r_{j_2}|r_{j_1}),\ldots,\psi(r_{j_n}|r_{j_m})$ defined over $\phi'(r_j)$ are disjoint due to Lemma 35 such that $\psi(r_{j_1}|r_j)$, $\psi(r_{j_2}|r_{j_1}),\ldots,\psi(r_{j_n}|r_{j_m})$ hold by Lemma 38 for any $j\in\mathfrak{L}$, $j_1\in\mathfrak{L}'(r_j)$, $j_2\in\mathfrak{L}'(r_{j_1}|r_j),\ldots,j_n\in\mathfrak{L}'(r_{j_m}|r_{j_l})$, thus $\phi'(r_j)$ is composed of $\psi(.)$ both disjoint and satisfied. Therefore, $\phi'(r_j)$ is *satisfiable*, and unsatisfiability of $\phi'_s(r_j)$ is *ignored* to verify $\nvDash\phi_s(r_j)$. Hence, Theorem 39 holds (see Proposition 30 and Note 31). Then, $\psi(r_i)\equiv\phi(r_i)$, since $\phi'(r_i)$ is satisfiable, and $\phi(r_i)=\psi(r_i)\wedge\phi'(r_i)$. Thus, Corollary 40 holds (see also Appendix A). ◀

Theorem 41 shows that any r_j incompatible remains incompatible, even if r_i is removed.

▶ **Theorem 41.** *If $\nvDash\varphi_{\tilde{s}}(r_j)$ for some $\tilde{s}$, then $\nvDash\varphi_s(r_j)$ for all $s>\tilde{s}$, even if $\neg r_i$ holds, $i\neq j$.*

Proof. See Note 27/28. $\nvDash\varphi_s(r_j)$ iff $\nvDash(\psi_s\wedge r_j)$ or $\nvDash\phi_s(r_j)$. Let $\nvDash(\psi_{\tilde{s}}\wedge r_j)$ for some $\tilde{s}$. Then, $\nvDash(\psi_s\wedge r_j)$ for all $s>\tilde{s}$, since $\psi_{\tilde{s}}\subseteq\psi_s$ due to `Remove` L:2. Let $\nvDash\phi_{\tilde{s}}(r_j)$ due to *solely* $x_i\wedge\overline{x}_i$. Then, $\overline{x}_i\vee x_i\Rightarrow\overline{r}_j$, thus $\overline{r}_j\in\psi_s$ for $s>\tilde{s}$. Hence, $\nvDash(\psi_s\wedge r_j)$ for all $s>\tilde{s}$. Assume that r_i is removed *before* r_j, that is, $\neg r_i$ holds by $\nvDash\varphi_{\check{s}}(r_i)$ for $\check{s}\leqslant\tilde{s}$. Then, $\neg r_i\Rightarrow\overline{r}_i$ and $\overline{r}_i\Rightarrow\overline{r}_j$, thus $\{\overline{r}_i,\overline{r}_j\}\subseteq\psi_s$ for $s>\tilde{s}$. Note that $\psi_{\check{s}}\subseteq\psi_{\tilde{s}}\subseteq\psi_s$. Hence, $\nvDash(\psi_s\wedge r_i\wedge r_j)$ for all $s>\tilde{s}$. If r_i is removed *after* r_j, i.e., $\neg r_i$ holds by $\nvDash\varphi_s(r_i)$ for $s>\tilde{s}$, then $\nvDash(\psi_s\wedge r_j\wedge r_i)$ for all $s>\tilde{s}$. ◀

▶ **Proposition 42.** *The time complexity of* `Scan` *is $O(mn^3)$.*

Proof. `OvrlEft`, and `Remove`, takes $4m$ steps by $\left(|\mathfrak{C}_*^{r_j}|\times|C_k|\right)+|\mathfrak{C}_*^{\overline{r}_j}|=3m+m$. `Scope` takes $n4m$ steps by $|\psi_s(r_j)|\times 4m$. Then, `Scan` takes $n^2 4m$ steps due to L:1-3 by $|\mathfrak{L}^\phi|\times|\psi_s|\times 4m$, as well as $8n^2m+8nm$ steps due to L:4-8 by $2|\mathfrak{L}^\phi|\times(4nm+4m)$. Also, the number of the scans is $\hat{s}\leqslant|\mathfrak{L}^\phi|$ due to `Remove` L:6. Therefore, the time complexity of `Scan` is $O(n^3m)$. ◀

▶ **Example 43.** $\varphi=\big\{\{\},\{x_3,x_4,\overline{x}_5\},\{x_3,x_6,\overline{x}_7\},\{x_4,x_6,\overline{x}_7\}\big\}$, i.e., $\psi=\emptyset$. Let `Scope`(x_3,ϕ) execute *first* in the *first* scan, which leads to the reductions below over ϕ due to x_3.

$$\begin{aligned}
\phi(x_3)=&\ (x_3\odot x_4\odot\overline{x}_5)\wedge(x_3\odot x_6\odot\overline{x}_7)\wedge(x_4\odot x_6\odot\overline{x}_7)\wedge x_3\\
x_3\Rightarrow&\ (x_3\wedge\overline{x}_4\wedge x_5)\wedge(x_3\wedge\overline{x}_6\wedge x_7)\wedge(x_4\odot x_6\odot\overline{x}_7)\wedge x_3\\
\overline{x}_4\Rightarrow&\ (x_3\wedge\overline{x}_4\wedge x_5)\wedge(x_3\wedge\overline{x}_6\wedge x_7)\wedge(\quad x_6\odot\overline{x}_7)\wedge x_3\\
\overline{x}_6\Rightarrow&\ (x_3\wedge\overline{x}_4\wedge x_5)\wedge(x_3\wedge\overline{x}_6\wedge x_7)\wedge(\quad\overline{x}_7)\wedge x_3
\end{aligned}$$

Since $\nvDash\big(\psi(x_3)=x_3\wedge\overline{x}_4\wedge x_5\wedge\overline{x}_6\wedge x_7\wedge\overline{x}_7\big)$, x_3 is incompatible, hence $\neg x_3\Rightarrow\overline{x}_3$, that is, $\overline{x}_3$ is necessary. Thus, $\varphi\rightarrow\varphi_2$ by $(x_3\odot x_4\odot\overline{x}_5)\rightarrowtail(x_4\odot\overline{x}_5)$ and $(x_3\odot x_6\odot\overline{x}_7)\rightarrowtail(x_6\odot\overline{x}_7)$. As a result, $\varphi_2=\overline{x}_3\wedge(x_4\odot\overline{x}_5)\wedge(x_6\odot\overline{x}_7)\wedge(x_4\odot x_6\odot\overline{x}_7)$. Let `Scope`$(x_5,\phi_2)$ execute next.

$$\begin{aligned}
\phi_2(x_5)=&\ (\quad x_4\odot\overline{x}_5)\wedge(\quad x_6\odot\overline{x}_7)\wedge(x_4\odot x_6\odot\overline{x}_7)\wedge x_5\\
x_5\Rightarrow&\ (\quad x_4\quad)\wedge(\quad x_6\odot\overline{x}_7)\wedge(x_4\odot x_6\odot\overline{x}_7)\wedge x_5\\
x_4\Rightarrow&\ (\quad x_4\quad)\wedge(\quad x_6\odot\overline{x}_7)\wedge(x_4\wedge\overline{x}_6\wedge x_7)\wedge x_5\\
\overline{x}_6\Rightarrow&\ (\quad x_4\quad)\wedge(\quad\overline{x}_7)\wedge(x_4\wedge\overline{x}_6\wedge x_7)\wedge x_5
\end{aligned}$$

Since $\nvDash\big(\psi_2(x_5)=x_4\wedge\overline{x}_7\wedge\overline{x}_6\wedge x_7\wedge\overline{x}_3\wedge x_5\big)$, x_5 is incompatible, hence $\neg x_5\Rightarrow\overline{x}_5$. Thus, $\varphi_2\rightarrow\varphi_3$ by $(x_4\odot\overline{x}_5)\searrow(\overline{x}_4\wedge\overline{x}_5)$, where $\varphi_3=\overline{x}_3\wedge\overline{x}_4\wedge\overline{x}_5\wedge(x_6\odot\overline{x}_7)\wedge(x_4\odot x_6\odot\overline{x}_7)$. Then, $\overline{x}_4$ leads to the next reduction by $(x_4\odot x_6\odot\overline{x}_7)\rightarrowtail(x_6\odot\overline{x}_7)$, and `Scan`$(\varphi_4)$ *terminates*. That is, $\hat{\varphi}=\hat{\psi}\wedge\hat{\phi}$, where $\hat{\psi}=\{\overline{x}_3,\overline{x}_4,\overline{x}_5\}$ and $\hat{\phi}=\big\{\{x_6,\overline{x}_7\}\big\}$, since $\varphi_4=\overline{x}_3\wedge\overline{x}_4\wedge\overline{x}_5\wedge(x_6\odot\overline{x}_7)$.

In Example 43, if $\texttt{Scope}\,(x_5, \phi)$ executes *first*, then $\psi(x_5) = x_5$ becomes the scope, and $\phi'(x_5) = (x_3 \odot x_4) \wedge (x_3 \odot x_6 \odot \overline{x}_7) \wedge (x_4 \odot x_6 \odot \overline{x}_7)$ becomes beyond the scope of x_5 over ϕ. Then, x_5 is compatible (in ϕ) due to Theorem 39, since $\psi(x_5)$ holds, while it is incompatible due to Proposition 30, since $\nvDash \phi'(x_5)$ holds. On the other hand, the fact that $\nvDash \phi'(x_5)$ holds is verified indirectly. That is, incompatibility of x_5 is checked by means of $\psi_s(x_5)$ for some s. Then, x_5 becomes incompatible (in ϕ_2), because $\nvDash \psi_2(x_5)$ holds, after $\varphi \to \varphi_2$ by removing x_3 from ϕ due to $\nvDash \psi(x_3)$. As a result, $\nvDash \phi'(x_5)$ holds due to $\neg x_3$. Thus, there exists no r_j such that $\nvDash \phi'(r_j)$, when the scan *terminates*, because $\psi(r_i)$ holds for all r_i in ϕ, hence $\psi(r_i|r_j)$ holds for all r_i in $\phi'(r_j)$, after each r_j is removed if $\nvDash \psi_s(r_j)$ (see also Figures 1-4).

3.4 Construction of a satisfying assignment by composing scopes

$\hat{\varphi} = \hat{\psi} \wedge \hat{\phi}$, when $\texttt{Scan}\,(\varphi_{\hat{s}})$ terminates. Let $\psi \coloneqq \hat{\psi}$ and $\phi \coloneqq \hat{\phi}$, i.e., $\mathfrak{L} \coloneqq \mathfrak{L}^{\hat{\phi}}$. Then, $\vDash_\alpha \phi$ holds by Corollary 40, where α is a satisfying assignment, and constructed by Algorithm 5 through any $(i_0, i_1, i_2, \ldots, i_m, i_n)$ over $\mathfrak{L}$ such that $\alpha = \{\psi(r_{i_0}), \psi(r_{i_1}|r_{i_0}), \psi(r_{i_2}|r_{i_1}), \ldots, \psi(r_{i_n}|r_{i_m})\}$. Thus, φ is decomposed into *disjoint* scopes $\psi, \psi(r_{i_0}), \psi(r_{i_1}|r_{i_0}), \psi(r_{i_2}|r_{i_1}), \ldots, \psi(r_{i_n}|r_{i_m})$ (see Note 28, and Lemma 34). Recall that any scope $\psi(.)$ denotes a minterm by Definition 4/5, and that $\texttt{Scope}\,(r_i, \phi)$ constructs $\psi(r_i)$ and $\phi'(r_i)$ to determine a satisfying assignment, unless φ collapses to a *unique* assignment, that is, unless $\hat{\varphi} = \alpha = \hat{\psi}$. See also Appendix A to determine a satisfying assignment without constructing $\psi(r_i|.)$ by $\texttt{Scope}\,\big(r_i, \phi'(.)\big)$.

Algorithm 5 ▷ Construction of a satisfying assignment α over ϕ, $\mathfrak{L} \coloneqq \mathfrak{L}^{\hat{\phi}}$ and $\phi \coloneqq \hat{\phi}$

Pick $j \in \mathfrak{L}$; ▷ The scope $\psi(r_i)$ and beyond the scope $\phi'(r_i)$ for all $i \in \mathfrak{L}$ are available initially
$\alpha \leftarrow \psi(r_j)$; $\mathfrak{L} \leftarrow \mathfrak{L} - \mathfrak{L}(r_j)$; $\phi \leftarrow \phi'(r_j)$;
repeat
  Pick $i \in \mathfrak{L}$; $\texttt{Scope}\,(r_i, \phi)$; ▷ It constructs $\psi(r_i|r_j)$ and $\phi'(r_i|r_j)$ with respect to $\phi'(r_j)$
  $\alpha \leftarrow \alpha \cup \psi(r_i)$; ▷ $\psi(r_i) \coloneqq \psi(r_i|r_j)$, because $\psi(r_i)$ is *unconditional* with respect to ϕ updated
  $\mathfrak{L} \leftarrow \mathfrak{L} - \mathfrak{L}(r_i)$; ▷ $\mathfrak{L} \leftarrow \mathfrak{L}'(r_i|r_j)$ due to the partition $\big\{\mathfrak{L}(r_j), \mathfrak{L}(r_i|r_j), \mathfrak{L}'(r_i|r_j)\big\}$ over $\mathfrak{L}$
  $\phi \leftarrow \phi'(r_i)$; ▷ $\phi'(r_i) \coloneqq \phi'(r_i|r_j)$, because $\phi'(r_i)$ is *unconditional* with respect to ϕ updated
until $\mathfrak{L} = \emptyset$
return α; ▷ $\psi(r_{i_n}|r_{i_m}) = \psi(r_{i_n}|r_j, r_{i_1}, \ldots, r_{i_m})$ (see also Appendix A)

▶ **Definition 44.** *Let $\phi = {}^1\phi \wedge {}^2\phi \wedge \cdots \wedge {}^l\phi$ such that ${}^1\phi, {}^2\phi, \ldots, {}^l\phi$ are disjoint, or independent formulas. That is, ${}^1\mathfrak{L} \cap {}^2\mathfrak{L} \cap \cdots \cap {}^l\mathfrak{L} = \emptyset$.*

▶ **Example 45.** Let ${}^1\phi = (x_1 \odot \overline{x}_2 \odot x_6) \wedge (x_3 \odot x_4 \odot \overline{x}_5) \wedge (x_3 \odot x_6 \odot \overline{x}_7) \wedge (x_4 \odot x_6 \odot \overline{x}_7)$, ${}^2\phi = (x_8 \odot x_9 \odot \overline{x}_{10})$, and ${}^3\phi = (x_{11} \odot \overline{x}_{12} \odot x_{13})$ to form $\varphi = {}^1\phi \wedge {}^2\phi \wedge {}^3\phi$ by Definition 44. Then, $\texttt{Scan}\,(\varphi_4)$ terminates, that is, φ is satisfiable. Thus, $\hat{\varphi} = \hat{\psi} \wedge \hat{\phi}$, where $\hat{\psi} = \overline{x}_3 \wedge \overline{x}_4 \wedge \overline{x}_5$ and $\hat{\phi} = (x_1 \odot \overline{x}_2 \odot x_6) \wedge (x_6 \odot \overline{x}_7) \wedge {}^2\phi \wedge {}^3\phi$ (see Example 43). Let $\psi \coloneqq \hat{\psi}$ and $\phi \coloneqq \hat{\phi}$, i.e., $\mathfrak{L} \coloneqq \mathfrak{L}^{\hat{\phi}}$. Hence, $\mathfrak{L}^{\psi} = \{3, 4, 5\}$, and $\mathfrak{L} = \{1, 2, \ldots, 13\} - \mathfrak{L}^{\psi}$. Then, a satisfying assignment α is determined by composing $\psi(r_i|r_j)$ constructed over $\phi'(r_j)$. The following shows some of the scopes $\psi(r_i)$ and beyond the scopes $\phi'(r_i)$, constructed over ϕ when the scan terminates.

$$
\begin{aligned}
\psi(x_1) &= x_1 \wedge x_2 \wedge \overline{x}_6 \wedge \overline{x}_7 & \& \quad \phi'(x_1) &= {}^2\phi \wedge {}^3\phi \\
\psi(x_2) &= x_2 & \& \quad \phi'(x_2) &= (x_1 \odot x_6) \wedge (x_6 \odot \overline{x}_7) \wedge {}^2\phi \wedge {}^3\phi \\
\psi(\overline{x}_2) &= \overline{x}_1 \wedge \overline{x}_2 \wedge \overline{x}_6 \wedge \overline{x}_7 & \& \quad \phi'(\overline{x}_2) &= {}^2\phi \wedge {}^3\phi \\
\psi(x_6) = \psi(x_7) &= \overline{x}_1 \wedge x_2 \wedge x_6 \wedge x_7 & \& \quad \phi'(x_6) = \phi'(x_7) &= {}^2\phi \wedge {}^3\phi \\
\psi(\overline{x}_6) = \psi(\overline{x}_7) &= \overline{x}_6 \wedge \overline{x}_7 & \& \quad \phi'(\overline{x}_6) = \phi'(\overline{x}_7) &= (x_1 \odot \overline{x}_2) \wedge {}^2\phi \wedge {}^3\phi \\
\psi(x_8) &= x_8 \wedge \overline{x}_9 \wedge x_{10} & \& \quad \phi'(x_8) &= (x_1 \odot \overline{x}_2 \odot x_6) \wedge (x_6 \odot \overline{x}_7) \wedge {}^3\phi \\
\psi(x_{11}) &= x_{11} \wedge x_{12} \wedge \overline{x}_{13} & \& \quad \phi'(x_{11}) &= (x_1 \odot \overline{x}_2 \odot x_6) \wedge (x_6 \odot \overline{x}_7) \wedge {}^2\phi
\end{aligned}
$$

▶ **Example 46.** A satisfying assignment α is constructed by an order of indices over $\mathfrak{L}$, $\mathfrak{L} = \{1,\dots,13\} - \mathfrak{L}^{\psi}$ (Example 45), such that $r_i \coloneqq x_i$ for any $\psi(r_i)$ throughout the construction. First, pick $6 \in \mathfrak{L}$. As a result, $\alpha \leftarrow \psi(x_6)$ and $\mathfrak{L} \leftarrow \mathfrak{L} - \mathfrak{L}(x_6)$, where $\psi(x_6) = \{\overline{x}_1, x_2, x_6, x_7\}$, $\mathfrak{L}(x_6) = \{1,2,6,7\}$, and $\mathfrak{L} \leftarrow \{8,9,10,11,12,13\}$. Then, pick 8, hence $\alpha \leftarrow \alpha \cup \psi(x_8|x_6)$, where $\psi(x_8|x_6) = \{x_8, \overline{x}_9, x_{10}\}$. Also, $\mathfrak{L} \leftarrow \mathfrak{L} - \mathfrak{L}(x_8|x_6)$, where $\mathfrak{L}(x_8|x_6) = \{8,9,10\}$, hence $\mathfrak{L} \leftarrow \{11,12,13\}$. Finally, pick 11. Therefore, $\alpha \leftarrow \alpha \cup \psi(x_{11}|x_6,x_8)$ such that $\mathfrak{L} \leftarrow \emptyset$, which indicates its termination. Note that $\texttt{Scope}\,(x_{11}, \phi'(x_8|x_6))$ constructs $\psi(x_{11}|x_6,x_8)$, in which $\phi'(x_8|x_6) = {}^3\phi$, and that $\mathfrak{L}'(x_{11}|x_6,x_8) = \emptyset$ iff $\mathfrak{L} \leftarrow \emptyset$. Note also that $\psi(x_8|x_6) = \psi(x_8)$ and $\psi(x_{11}|x_6,x_8) = \psi(x_{11})$, since ${}^1\phi$, ${}^2\phi$ and ${}^3\phi$ are disjoint by Definition 44. Consequently, Algorithm 5 constructs $\alpha = \{\psi(x_6), \psi(x_8|x_6), \psi(x_{11}|x_6,x_8)\}$. Note that φ is *decomposed* into ψ, $\psi(x_6)$, $\psi(x_8|x_6)$, and $\psi(x_{11}|x_6,x_8)$, which are *disjoint* (see also Note 29 and Lemma 34).

▶ **Example 47.** Let $(2,1,8,11)$ be another order of indices in Example 45. This order leads to the assignment $\{\psi, \psi(x_2), \psi(x_1|x_2), \psi(x_8|x_2,x_1), \psi(x_{11}|x_2,x_1,x_8)\}$ for φ. This assignment corresponds to the partition $\{\mathfrak{L}^{\psi}, \{2\}, \{1,6,7\}, \{8,9,10\}, \{11,12,13\}\}$, where $\mathfrak{L}^{\psi} = \{3,4,5\}$ (see also Note 28 and Lemma 34). Note that the scope $\psi(x_1)$ is constructed over ϕ, and the conditional scope $\psi(x_1|x_2)$ is constructed over $\phi'(x_2)$, where $\phi \supseteq \phi'(x_2)$. Recall that $\phi \coloneqq \hat{\phi}$. Hence, $\psi(x_1) \vDash \psi(x_1|x_2)$, in which $\psi(x_1) = x_1 \wedge x_2 \wedge \overline{x}_6 \wedge \overline{x}_7$, while $\psi(x_1|x_2) = x_1 \wedge \overline{x}_6 \wedge \overline{x}_7$. Moreover, $\psi(x_8) \vDash \psi(x_8|x_2,x_1)$ due to $\phi \supseteq \phi'(x_1|x_2)$, and $\psi(x_{11}) \vDash \psi(x_{11}|x_2,x_1,x_8)$ due to $\phi \supseteq \phi'(x_8|x_2,x_1)$, where $\phi'(x_1|x_2) = {}^2\phi \wedge {}^3\phi$ and $\phi'(x_8|x_2,x_1) = {}^3\phi$ (see Lemmas 36-38).

3.5 An Illustrative Example

This section illustrates $\texttt{Scan}\,(\varphi_s)$. Let $\varphi = \phi = (x_1 \odot \overline{x}_3) \wedge (x_1 \odot \overline{x}_2 \odot x_3) \wedge (x_2 \odot \overline{x}_3)$, which is adapted from Esparza [2], and denotes a general formula by Definition 15. Note that $C_1 = \{x_1, \overline{x}_3\}$, $C_2 = \{x_1, \overline{x}_2, x_3\}$, and $C_3 = \{x_2, \overline{x}_3\}$. Hence, $\mathfrak{C} = \{1,2,3\}$, and $\mathfrak{L} = \mathfrak{L}^{\phi} = \{1,2,3\}$.

$\texttt{Scan}\,(\varphi)$: There exists no conjunct in (the initial formula) φ. That is, ψ is empty (L:1). Recall that $\varphi \coloneqq \varphi_1$, and that $r_i \in \{x_i, \overline{x}_i\}$. Recall also that *nontrivial* incompatibility of r_i is checked (L:4-8) via $\texttt{Scope}\,(r_i, \phi)$. Moreover, the order of incompatibility check is arbitrary (incompatibility is monotonic) by Theorem 41. Let $\texttt{Scope}\,(x_1, \phi)$ execute due to $\texttt{Scan}$ L:6.

$\texttt{Scope}\,(x_1, \phi)$: Since $\psi(x_1) \supseteq \{x_3, \overline{x}_3\}$, x_1 is incompatible *nontrivially* (see Example 25). Thus, $\overline{x}_1$ becomes necessary (a conjunct). Then, $\texttt{Remove}\,(x_1, \phi)$ executes due to $\texttt{Scan}$ L:6.

$\texttt{Remove}\,(x_1, \phi)$: $\mathfrak{C}^{\overline{x}_1} = \emptyset$ by $\texttt{OvrlEft}$ L:1. $\mathfrak{C}^{x_1} = \{1,2\}$, thus $\phi^{x_1} = (x_1 \odot \overline{x}_3) \wedge (x_1 \odot \overline{x}_2 \odot x_3)$ by $\texttt{OvrlEft}$ L:7. As a result, $\tilde{\psi}(\overline{x}_1) = \{\overline{x}_3\}$ & $\tilde{\phi}(\neg x_1) = \{\{\}, \{\overline{x}_2, x_3\}\}$, the effects of $\overline{x}_1$ and $\neg x_1$. Note that $C_1 \leftarrow \emptyset$. Then, $\psi_2 \leftarrow \psi \cup \{\overline{x}_1\} \cup \tilde{\psi}(\overline{x}_1)$ ($\texttt{Remove}$ L:2), and $\mathfrak{L}^{\phi} \leftarrow \mathfrak{L}^{\phi} - \{1\}$ and $\mathfrak{L}^{\psi} \leftarrow \mathfrak{L}^{\psi} \cup \{1\}$ (L:4). Also, $\phi_2 \leftarrow \tilde{\phi}(\neg x_1) \wedge \phi'$, where $\tilde{\phi}(\neg x_1) = (\overline{x}_2 \odot x_3)$ and $\phi' = (x_2 \odot \overline{x}_3)$ (L:5). As a result, $\psi_2 = \overline{x}_1 \wedge \overline{x}_3$, and $\phi_2 = (\overline{x}_2 \odot x_3) \wedge (x_2 \odot \overline{x}_3)$. Note that $C_1 = \{\overline{x}_2, x_3\}$ and $C_2 = \{x_2, \overline{x}_3\}$. Consequently, $\varphi_2 = \psi_2 \wedge \phi_2$, and $\texttt{Scan}\,(\varphi_2)$ executes due to $\texttt{Remove}$ L:6.

$\texttt{Scan}\,(\varphi_2)$: $\mathfrak{C}_2 = \{1,2\}$ and $\mathfrak{L}^{\phi} = \{2,3\}$ hold in ϕ_2. Then, $\{x_2, \overline{x}_2\} \cap \psi_2 = \emptyset$ for $2 \in \mathfrak{L}^{\phi}$, while $\overline{x}_3 \in \psi_2$ for $3 \in \mathfrak{L}^{\phi}$ (L:1). As a result, $\overline{x}_3$ is *necessary* for satisfying φ_2, hence $\overline{x}_3 \Rightarrow \neg x_3$, that is, x_3 is incompatible *trivially*. Then, $\texttt{Remove}\,(x_3, \phi_2)$ executes due to $\texttt{Scan}$ L:2.

$\texttt{Remove}\,(x_3, \phi_2)$: $\mathfrak{C}_2^{\overline{x}_3} = \{2\}$, thus $\phi_2^{\overline{x}_3} = (x_2 \odot \overline{x}_3)$, and $\mathfrak{C}_2^{x_3} = \{1\}$, thus $\phi_2^{x_3} = (\overline{x}_2 \odot x_3)$. As a result, $\tilde{\psi}_2(\overline{x}_3) = \{\overline{x}_2\} \cup \{\overline{x}_2\}$ & $\tilde{\phi}_2(\neg x_3) = \{\{\}\}$, because $C_1 = \{\overline{x}_2\}$ consists in $\tilde{\psi}_2(\overline{x}_3)$, rather than in $\tilde{\phi}_2(\neg x_3)$ (see $\texttt{OvrlEft}$ L:9). Hence, $\psi_3 \leftarrow \psi_2 \cup \{\overline{x}_3\} \cup \tilde{\psi}_2(\overline{x}_3)$, $\mathfrak{L}^{\phi} \leftarrow \mathfrak{L}^{\phi} - \{3\}$, and $\mathfrak{L}^{\psi} \leftarrow \mathfrak{L}^{\psi} \cup \{3\}$, i.e., $\mathfrak{L}^{\phi} = \{2\}$. Therefore, $\phi_3 = \{\{\}\}$, thus $\mathfrak{C}_3 = \emptyset$, and $\psi_3 = \overline{x}_1 \wedge \overline{x}_3 \wedge \overline{x}_2$.

$\texttt{Scan}\,(\varphi_3)$: $\overline{x}_2 \in \psi_3$ for $2 \in \mathfrak{L}^{\phi}$ over ϕ_3. Then, $\texttt{Remove}\,(x_2, \phi_3)$ executes due to $\texttt{Scan}$ L:2.

$\texttt{Remove}\,(x_2, \phi_3)$: $\tilde{\psi}_3(\overline{x}_2) = \emptyset$ & $\tilde{\phi}_3(\neg x_2) = \{\{\}\}$ due to $\texttt{OvrlEft}\,(\overline{x}_2, \phi_3)$, because $\mathfrak{C}_3^{\overline{x}_2} = \emptyset$ and $\mathfrak{C}_3^{x_2} = \emptyset$, since $\mathfrak{C}_3 = \emptyset$. Hence, $\mathfrak{L}^{\phi} \leftarrow \{2\} - \{2\}$ and $\phi_4 \leftarrow \phi_3$. Then, $\texttt{Scan}\,(\varphi_4)$ executes.

$\texttt{Scan}\,(\varphi_4)$ *terminates*: $\hat{\varphi} = \hat{\psi} = \overline{x}_1 \wedge \overline{x}_3 \wedge \overline{x}_2$ (L:9), and φ collapses to a unique assignment.

Let $\texttt{Scope}\,(x_3, \phi)$ execute *before* $\texttt{Scope}\,(x_1, \phi)$ due to $\texttt{Scan}$ L:6 (see Theorem 41).

$\texttt{Scope}\,(x_3, \phi)$: $\psi(x_3) \leftarrow \{x_3\}$ and $\phi_* \leftarrow \phi$ (L:1). Then, $\mathfrak{C}_*^{x_3} = \{2\}$ due to $\texttt{OvrlEft}\,(x_3, \phi_*)$ L:1, hence $\phi_*^{x_3} = (x_1 \odot \overline{x}_2 \odot x_3)$. As a result, $c_2 \leftarrow \{\overline{x}_1, x_2\}$ and $\tilde{\psi}_*(x_3) \leftarrow \tilde{\psi}_*(x_3) \cup c_2$ (L:3,5). Moreover, $\mathfrak{C}_*^{\overline{x}_3} = \{1, 3\}$ (L:7), hence $\phi_*^{\overline{x}_3} = (x_1 \odot \overline{x}_3) \wedge (x_2 \odot \overline{x}_3)$. Then, $C_1 \leftarrow \{x_1, \overline{x}_3\} - \{\overline{x}_3\}$, $\tilde{\psi}_*(x_3) \leftarrow \tilde{\psi}_*(x_3) \cup C_1$, and $C_1 \leftarrow \emptyset$. Likewise, $C_3 \leftarrow \{x_2, \overline{x}_3\} - \{\overline{x}_3\}$, $\tilde{\psi}_*(x_3) \leftarrow \tilde{\psi}_*(x_3) \cup C_3$, and $C_3 \leftarrow \emptyset$ ($\texttt{OvrlEft}$ L:8-9). Consequently, $\tilde{\psi}_*(x_3) \leftarrow \{\overline{x}_1, x_2, x_1\}$ & $\tilde{\phi}_*(\neg x_3) \leftarrow \phi_*^{\overline{x}_3}$ (L:11). Note that $\phi_*^{\overline{x}_3} = \{\{\}, \{\}\}$, since $C_1 = C_3 = \emptyset$. Then, $\psi(x_3) \leftarrow \psi(x_3) \cup \{x_3\} \cup \tilde{\psi}_*(x_3)$ due to $\texttt{Scope}$ L:4, hence $\psi(x_3) = \{x_3, \overline{x}_1, x_2, x_1\}$. Since $\psi(x_3) \supseteq \{\overline{x}_1, x_1\}$ (L:5), x_3 is incompatible *nontrivially*, i.e., $x_3 \Rightarrow \overline{x}_1 \wedge x_1$ and $\neg x_3 \Rightarrow \overline{x}_3$. Then, $\texttt{Remove}\,(x_3, \phi)$ executes due to $\texttt{Scan}$ L:6.

$\texttt{Remove}\,(x_3, \phi)$: $\phi^{\overline{x}_3} = (x_1 \odot \overline{x}_3) \wedge (x_2 \odot \overline{x}_3)$ due to $\mathfrak{C}^{\overline{x}_3} = \{1, 3\}$, and $\phi^{x_3} = (x_1 \odot \overline{x}_2 \odot x_3)$ due to $\mathfrak{C}^{x_3} = \{2\}$. Then, $\texttt{OvrlEft}\,(\overline{x}_3, \phi)$ returns $\tilde{\psi}(\overline{x}_3) = \{\overline{x}_1, \overline{x}_2\}$ & $\tilde{\phi}(\neg x_3) = \{\{x_1, \overline{x}_2\}\}$ ($\texttt{Remove}$ L:1), $\psi_2 \leftarrow \psi \cup \{\overline{x}_3\} \cup \tilde{\psi}(\overline{x}_3)$ (L:2), and $\mathfrak{L}^\phi \leftarrow \mathfrak{L}^\phi - \{3\}$ and $\mathfrak{L}^\psi \leftarrow \mathfrak{L}^\psi \cup \{3\}$ (L:4). As a result, $\psi_2 = \overline{x}_3 \wedge \overline{x}_1 \wedge \overline{x}_2$. Moreover, $\phi_2 \leftarrow \tilde{\phi}(\neg x_3) \wedge \phi'$ (L:5), in which $\tilde{\phi}(\neg x_3) = (x_1 \odot \overline{x}_2)$ and ϕ' is empty. Therefore, $\varphi_2 = \psi_2 \wedge \phi_2$. Note that $C_1 = \{x_1, \overline{x}_2\}$, hence $\mathfrak{C}_2 = \{1\}$. Recall that $\mathfrak{L}^\phi = \{1, 2\}$, and that $\mathfrak{L}^\psi = \{3\}$. Then, $\texttt{Scan}\,(\varphi_2)$ executes due to $\texttt{Remove}\,(x_3, \phi)$ L:6.

$\texttt{Scan}\,(\varphi_2)$: $\mathfrak{L}^\phi = \{1, 2\}$ such that $\overline{x}_2 \in \psi_2$ and $\overline{x}_1 \in \psi_2$. Thus, $\overline{x}_2$ and $\overline{x}_1$ are *necessary*, hence x_2 and x_1 are incompatible *trivially*. Then, $\texttt{Remove}\,(x_1, \phi_2)$ and $\texttt{Remove}\,(x_2, \phi_2)$ execute.

The fact that the order of incompatibility check is arbitrary (Theorem 41) is illustrated as follows. $\texttt{Scope}\,(x_3, \phi)$ returns x_3 is incompatible *nontrivially*, since $x_3 \Rightarrow \overline{x}_1 \wedge x_1$. Therefore, $\neg\overline{x}_1 \vee \neg x_1 \Rightarrow \neg x_3$, hence $x_1 \vee \overline{x}_1 \Rightarrow \overline{x}_3$. Then, $\overline{x}_3 \Rightarrow \overline{x}_1$ due to $C_1 = (x_1 \odot \overline{x}_3)$, and $\overline{x}_1 \Rightarrow \neg x_1$. Thus, x_1 is *still* incompatible, but trivially $\big($cf. $\texttt{Scope}\,(x_1, \phi)\big)$, even if $\neg x_3$ holds. That is, x_1 the *nontrivial* incompatible in ϕ due to $x_1 \Rightarrow \overline{x}_3 \wedge x_3$, i.e., $\neg\overline{x}_3 \vee \neg x_3 \Rightarrow \neg x_1$, is incompatible *trivially* in ψ_2 due to $\overline{x}_1 \Rightarrow \neg x_1$. See $\texttt{Scan}\,(\varphi_2)$ above. Also, since $x_3 \notin C_k$ and $\overline{x}_3 \notin C_k$ in ϕ_s for any $s \geqslant 2$, $\nvDash \varphi_s(x_3)$ for all $s \geqslant 2$, even if any r_i is removed from some C_k in ϕ_s, $s \geqslant 2$.

4 Conclusion

X3SAT has proved to be effective to show $\mathbf{P} = \mathbf{NP}$. A polynomial time algorithm checks unsatisfiability of $\phi(r_i)$ such that $\nvDash \phi(r_i)$ iff $\psi_s(r_i)$ involves $x_j \wedge \overline{x}_j$ for some s. Thus, $\phi(r_i)$ reduces to $\psi(r_i)$. $\psi(r_i)$ denotes a conjunction of literals that are *true*, since each r_j such that $\nvDash \psi_s(r_j)$ is removed from ϕ. Hence, ϕ is satisfiable iff $\psi(r_i)$ is satisfied for any $r_i \in \{x_i, \overline{x}_i\}$. Thus, it is *easy* to verify satisfiability of ϕ via satisfiability of $\psi(x_1), \psi(\overline{x}_1), \ldots, \psi(x_n), \psi(\overline{x}_n)$.

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A Proof of Theorem 39/40

This section gives a rigorous proof of Theorem 39/40. Recall that the φ_s scan is *interrupted* iff ψ_s involves $x_i \wedge \overline{x}_i$ for some i and s, that is, φ is unsatisfiable, which is trivial to verify. Recall also that the $\varphi_{\hat{s}}$ scan *terminates* iff $\psi_{\hat{s}}(r_i) = \mathbf{T}$ for any $i \in \mathfrak{L}^\phi$, $r_i \in \{x_i, \overline{x}_i\}$. Moreover, $\hat{\varphi} = \hat{\psi} \wedge \hat{\phi}$ such that $\hat{\psi} = \mathbf{T}$ (see $\texttt{Scan}$ L:9 and Note 29). Therefore, when the scan terminates, satisfiability of $\hat{\phi}$ is to be proved, which is addressed in this section. Let $\phi := \hat{\phi}$, i.e., $\mathfrak{L} := \mathfrak{L}^{\hat{\phi}}$.

▶ **Theorem 48** (cf. 39-40/Claim 1). *These statements are equivalent for any $i \in \mathfrak{L}$: a) $\nvDash \phi(r_i)$ iff $\nvDash \psi_s(r_i)$ for some s. b) $r_i \vDash \psi(r_i)$. c) $\vDash_\alpha \phi$ by $\alpha = \{\psi(r_{i_0}), \psi(r_{i_1}|r_{i_0}), \ldots, \psi(r_{i_n}|r_{i_m})\}$.*

Proof. We will show $a \Rightarrow b$, $b \Rightarrow c$, and $c \Rightarrow a$ (see Kenneth H. Rosen, Discrete Mathematics and its Applications, 7E, pg. 88). Firstly, $a \Rightarrow b$ holds, because a holds by assumption (see Note 31), and b holds by Lemma 23. Next, we will show $b \Rightarrow c$. We do this by showing that satisfiability of ϕ is *preserved* throughout the assignment α construction, where $\alpha = \{\psi(r_{i_0}), \psi(r_{i_1}|r_{i_0}), \ldots, \psi(r_{i_n}|r_{i_m})\}$, because any *partial* assignment $\psi(r_i|r_j)$ is constructed *arbitrarily* through consecutive steps having the Markov property. Thus, construction of $\psi(r_i|r_j)$ in the next step is independent from the preceding steps, and depends only upon $\psi(r_j|r_k)$ in the present step (see also Lemma 34). The construction process is specified below.

Step 0: Pick any r_{i_0} in ϕ. Then, $r_{i_0} \vDash \psi(r_{i_0})$ by Lemma 23. Also, r_{i_0} partitions $\mathfrak{L}$ into $\mathfrak{L}(r_{i_0})$ and $\mathfrak{L}'(r_{i_0})$. Note that $i_0 \in \mathfrak{L}$ and $i_0 \in \mathfrak{L}(r_{i_0})$. Hence, $i_0 \notin \mathfrak{L}'(r_{i_0})$ by Lemma 33. Therefore, $\phi(r_{i_0}) = \psi(r_{i_0}) \wedge \phi'(r_{i_0})$ in Step 0. Then, pick an *arbitrary* r_{i_1} in $\phi'(r_{i_0})$ for Step 1.

Step 1: $\mathfrak{L}(r_{i_0}) \cap \mathfrak{L}'(r_{i_0}) = \emptyset$ due to Step 0. Then, $r_{i_1} \vDash \psi(r_{i_1})$ by Lemma 23, as well as $\psi(r_{i_1}) \vDash \psi(r_{i_1}|r_{i_0})$ by Lemma 37. Also, r_{i_1} partitions $\mathfrak{L}'(r_{i_0})$ into $\mathfrak{L}(r_{i_1}|r_{i_0})$ and $\mathfrak{L}'(r_{i_1}|r_{i_0})$. Thus, $\mathfrak{L}(r_{i_0}) \cap \mathfrak{L}(r_{i_1}|r_{i_0}) = \emptyset$, since $\mathfrak{L}'(r_{i_0}) \supseteq \mathfrak{L}(r_{i_1}|r_{i_0})$. As a result, $\mathfrak{L}$ is partitioned into $\mathfrak{L}(r_{i_0})$, $\mathfrak{L}(r_{i_1}|r_{i_0})$, and $\mathfrak{L}'(r_{i_1}|r_{i_0})$ by r_{i_0} and r_{i_1}. Thus, $\psi(r_{i_0})$ and $\psi(r_{i_1}|r_{i_0})$ are *disjoint*, as well as *true*. Therefore, $\psi(r_{i_0}) \wedge \psi(r_{i_1}|r_{i_0}) = \mathbf{T}$, and $\phi(r_{i_0}, r_{i_1}) = \psi(r_{i_0}) \wedge \psi(r_{i_1}|r_{i_0}) \wedge \phi'(r_{i_1}|r_{i_0})$.

Step 2: The preceding steps have partitioned $\mathfrak{L}$ into $\mathfrak{L}(r_{i_0}) \cup \mathfrak{L}(r_{i_1}|r_{i_0})$ and $\mathfrak{L}'(r_{i_1}|r_{i_0})$. Then, $r_{i_2} \vDash \psi(r_{i_2})$ by Lemma 23, as well as $\psi(r_{i_2}) \vDash \psi(r_{i_2}|r_{i_1})$ by Lemma 37/38. Also, r_{i_2} in $\phi'(r_{i_1}|r_{i_0})$ partitions $\mathfrak{L}'(r_{i_1}|r_{i_0})$ into $\mathfrak{L}(r_{i_2}|r_{i_1})$ and $\mathfrak{L}'(r_{i_2}|r_{i_1})$, i.e., $\mathfrak{L}'(r_{i_1}|r_{i_0}) \supseteq \mathfrak{L}(r_{i_2}|r_{i_1})$. Then, $\big(\mathfrak{L}(r_{i_0}) \cup \mathfrak{L}(r_{i_1}|r_{i_0})\big) \cap \mathfrak{L}(r_{i_2}|r_{i_1}) = \emptyset$, thus $\psi(r_{i_0}) \wedge \psi(r_{i_1}|r_{i_0})$ and $\psi(r_{i_2}|r_{i_1})$ are *disjoint*, as well as *true*. Therefore, $\phi(r_{i_0}, r_{i_1}, r_{i_2}) = \psi(r_{i_0}) \wedge \psi(r_{i_1}|r_{i_0}) \wedge \psi(r_{i_2}|r_{i_1}) \wedge \phi'(r_{i_2}|r_{i_1})$, in which $\psi(r_{i_0}) \wedge \psi(r_{i_1}|r_{i_0}) \wedge \psi(r_{i_2}|r_{i_1}) = \mathbf{T}$. Note that $\alpha \supseteq \{\psi(r_{i_0}), \psi(r_{i_1}|r_{i_0}), \psi(r_{i_2}|r_{i_1})\}$, and that $\mathfrak{L}$ is partitioned into $\mathfrak{L}(r_{i_0})$, $\mathfrak{L}(r_{i_1}|r_{i_0})$, $\mathfrak{L}(r_{i_2}|r_{i_1})$, and $\mathfrak{L}'(r_{i_2}|r_{i_1})$ such that $\mathfrak{L}'(r_{i_2}|r_{i_1}) \neq \emptyset$.

Step n: r_{i_n} partitions $\mathfrak{L}'(r_{i_m}|r_{i_l})$ into $\mathfrak{L}(r_{i_n}|r_{i_m})$ and $\mathfrak{L}'(r_{i_n}|r_{i_m})$ such that $\mathfrak{L}'(r_{i_n}|r_{i_m}) = \emptyset$. $\mathfrak{L}(r_{i_0}) \cup \mathfrak{L}(r_{i_1}|r_{i_0}) \cup \cdots \cup \mathfrak{L}(r_{i_m}|r_{i_l})$ and $\mathfrak{L}'(r_{i_m}|r_{i_l})$, hence $\mathfrak{L}(r_{i_n}|r_{i_m})$, form a partition of $\mathfrak{L}$. Therefore, $\psi(r_{i_0}) \wedge \psi(r_{i_1}|r_{i_0}) \wedge \cdots \wedge \psi(r_{i_m}|r_{i_l})$ and $\psi(r_{i_n}|r_{i_m})$ are *disjoint*, as well as *true*. That is, $\phi(r_{i_0}, r_{i_1}, \ldots, r_{i_m}, r_{i_n}) = \psi(r_{i_0}) \wedge \psi(r_{i_1}|r_{i_0}) \wedge \cdots \wedge \psi(r_{i_m}|r_{i_l}) \wedge \psi(r_{i_n}|r_{i_m})$ is satisfied.

Thus, ϕ is composed of $\psi(.)$ *disjoint* and *satisfied*, hence ϕ is satisfiable, and $b \Rightarrow c$ holds. Finally, we show $c \Rightarrow a$. r_i transforms ϕ into $\psi(r_i) \wedge \phi'(r_i)$. Then, $\phi \equiv \psi(r_i) \wedge \phi'(r_i)$, where ϕ and $\psi(r_i)$ are *satisfiable*, and $\psi(r_i)$ and $\phi'(r_i)$ are *disjoint*. Thus, $\phi'(r_i)$ is satisfiable. Hence, unsatisfiability of $\psi_s(r_i)$ for some s is necessary and sufficient for $\nvDash \phi_s(r_i)$ for any s. ◀

▶ Note. The assignment α construction is driven by partitioning the set $\mathfrak{L}'(.)$ such that $\mathfrak{L} \leftarrow \mathfrak{L} - \mathfrak{L}(r_{i_0})$ in Step 1, and $\mathfrak{L} \leftarrow \mathfrak{L} - \mathfrak{L}(r_{i_{n-1}}|r_{i_{n-2}})$ for $i_n \in \mathfrak{L}'(r_{i_{n-1}}|r_{i_{n-2}})$ in Step $n \geqslant 2$.

▶ Note. $\psi(r_i) \equiv \phi(r_i)$ by Theorem 48. Thus, the formula $\phi = \bigwedge_{k \in \mathfrak{C}} C_k$ transforms into the formula $\phi' = \bigwedge_{i \in \mathfrak{L}} \mathcal{C}_i$, where $C_k = (r_i \odot r_j \odot r_v)$ and $\mathcal{C}_i = \big(\psi(x_i) \oplus \psi(\overline{x}_i)\big)$. See also Note 29.

▶ Note (Construction of α). In order to form a partition over the set ϕ, α is constructed such that $\psi(r_{i_1}|r_{i_0}) = \psi(r_{i_1}) - \psi(r_{i_0})$, and $\psi(r_{i_n}|r_{i_{n-1}}) = \psi(r_n) - \big(\psi(r_{i_0}) \cup \cdots \cup \psi(r_{i_{n-1}}|r_{i_{n-2}})\big)$ for $n \geqslant 2$. On the other hand, if the construction involves no set partition, then $\alpha = \bigcup \psi(r_i)$ for $i = (i_0, i_1, \ldots, i_n)$, where $i_0 \in \mathfrak{L}$, $i_1 \in \mathfrak{L}'(r_{i_0}), \ldots, i_n \in \mathfrak{L}'(r_{i_m}|r_{i_l})$, thus $r_{i_0} \prec r_{i_1} \prec \cdots \prec r_{i_n}$. Note that there is no need to construct $\phi'(r_i)$ in `Scan`/`Scope` L:9 (cf. Algorithm 5).

For instance, if Example 45 involves no set partition, then $\alpha = \{\psi(\overline{x}_7), \psi(x_2), \psi(x_1)\}$, in which $\psi(\overline{x}_7) = \{\overline{x}_7, \overline{x}_6\}$, $\psi(x_2) = \{x_2\}$, and $\psi(x_1) = \{x_1, x_2, \overline{x}_7, \overline{x}_6\}$. Also, $\overline{x}_7 \prec x_2 \prec x_1$ due to $x_2 \in \phi'(\overline{x}_7)$ and $x_1 \in \phi'(x_2|\overline{x}_7)$. Moreover, $\psi(\overline{x}_7)$, $\psi(x_2|\overline{x}_7)$, and $\psi(x_1|x_2)$ form a partition over the set ϕ, where $\psi(x_2|\overline{x}_7) = \psi(x_2) - \psi(\overline{x}_7)$ and $\psi(x_1|x_2) = \psi(x_1) - \big(\psi(x_2|\overline{x}_7) \cup \psi(\overline{x}_7)\big)$. As a result, $\alpha = \phi(\overline{x}_7, x_2, x_1) = \{\overline{x}_7, \overline{x}_6\} \cup \{x_2\} \cup \{x_1\}$ such that $\{\overline{x}_7, \overline{x}_6\} \cap \{x_2\} \cap \{x_1\} = \emptyset$.